Fully Coupled Pauli-Fierz Systems at Zero and Positive Temperature

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Contents

1 Introduction

The purpose of these notes is to give a fairly narrow but thorough introduction to the spectral analysis of Hamiltonians and standard Liouvilleans describing finite dimensional small systems linearly coupled to a scalar massless field or reservoir. The Hamiltonians describe the system at zero temperature, and the standard Liouvillean implements unitarily the dynamics of the system at positive temperature.

We focus our attention on results valid at arbitrary coupling strength and whose proofs are purely operator theoretic, i.e. for the standard Liouvillean does not make use of the underlying modular structure. This means that important structure results at positive temperature that does not seem to have a purely operator theoretic proof will only be reviewed.

In the rest of the introduction we will assume some familiarity with quantum dynamical systems, standard representations of von Neumann algebras, and in particular the bosonic Weyl-algebra and its positive temperature standard form, given in terms of Araki-Woods fields. We refer the reader to the three review papers [8, 48, 57] written by Attal, Merkli, and Pillet in connection with an earlier summer school held in Grenoble in 2003. They combine to give an excellent introduction to the subject. As soon as the problems we aim for have been cast in spectral terms at the end of the introduction, these topics will not play a central role anymore.

1.1 The Small Quantum System

Our small quantum system lives in the finite dimensional Hilbert space $\mathcal{K} = \mathbb{C}^\nu$. The associated observable algebra is taken to be $\mathfrak{M}_{\mathrm{p}} = \mathcal{B}(\mathcal{K}) = \mathrm{M}_\nu(\mathbb{C})$, the ν^2-dimensional space of $\nu \times \nu$ matrices. The subscript p signals that the algebra belongs to the particle system.

As a Hamiltonian we take a diagonal matrix

$$K = \mathrm{diag}\{E_1, E_2, \ldots, E_\nu\},$$

where the eigenvalues $E_1 \leq E_1 \leq \cdots \leq E_\nu$ are real numbers. It suffices for K to be merely self-adjoint, but we can obviously always choose a basis in which K has the form considered here.

States ω on the algebra $\mathfrak{M}_{\mathrm{p}}$ are identified with density matrices ρ, i.e. self-adjoint matrices with $0 \leq \rho \leq 1$ and $\mathrm{Tr}(\rho) = 1$. The associated state is

$$\omega(T) = \mathrm{Tr}(\rho T).$$

A state is called a vector state if ρ is a rank one projection, i.e. $\rho = |u\rangle\langle u|$ and hence $\omega(T) = \langle u, Tu\rangle$. Here u is a normalized vector.

The Hamiltonian K generates a dynamics on the algebra as well as on the states ω as follows: $T \to T_t = \mathrm{e}^{\mathrm{i}tK} T \mathrm{e}^{-\mathrm{i}tK}$ and $\omega \to \omega_t$ defined by $\omega_t(T) = \omega(T_t)$.

The set of states invariant under the dynamics can be identified with density matrices of the form

$$\sum \alpha_j P_j,$$

where $0 \leq \alpha_j \leq 1$, P_j are orthogonal projections onto (subspaces of) eigenspaces of K, and $\sum \alpha_j \mathrm{Tr}(P_j) = 1$.

At inverse temperature β the so-called Gibbs, hence β-KMS, invariant state on $\mathfrak{M}_{\mathrm{p}}$ is

$$\omega_\beta^{\mathrm{p}}(T) = \mathrm{Tr}(\rho_\beta T), \qquad \rho_\beta = \frac{\mathrm{e}^{-\beta K}}{\mathrm{Tr}(\mathrm{e}^{-\beta K})}.$$

At zero temperature, or $\beta = +\infty$, this becomes $\rho_\infty = \nu_0^{-1} P_0$, where P_0 is the orthogonal projection onto the span of ground states, and $\nu_0 = \mathrm{Tr}(P_0)$ is the multiplicity of the ground state. If the ground state is non-degenerate, $\omega_\infty^{\mathrm{p}}$ is a vector state.

As a last item we wish to discuss the GNS representation of the algebra $\mathfrak{M}_{\mathrm{p}}$ with respect to the Gibbs state $\omega_\beta^{\mathrm{p}}$, which in fact coincides with its standard representation.

Typically it is introduced as left multiplication on the algebra itself, but through the identification $|u\rangle\langle v| \to u \otimes \bar{v} \in \mathcal{K} \otimes \mathcal{K}$ we pass straight to a representation in terms of operators on $\mathcal{K} \otimes \mathcal{K}$.

We have the left representation $\pi_{\mathrm{l}}^{\mathrm{p}}$, which serves as the GNS representation, and the (conjugate linear) right representation $\pi_{\mathrm{r}}^{\mathrm{p}}$ defined by

$$\pi_{\mathrm{l}}^{\mathrm{p}}(T) = T \otimes \mathbb{1}_{\mathcal{K}} \quad \text{and} \quad \pi_{\mathrm{r}}^{\mathrm{p}}(T) = \mathbb{1}_{\mathcal{K}} \otimes \overline{T}.$$

They are intertwined by the conjugate linear modular conjugation $J_{\mathrm{p}} \colon \mathcal{K} \otimes \mathcal{K} \to \mathcal{K} \otimes \mathcal{K}$ defined by

$$J_{\mathrm{p}}(u \otimes v) = \bar{u} \otimes \bar{v}, \tag{1.1}$$

and extended by linearity to $\mathcal{K} \otimes \mathcal{K}$. That is, $\pi_{\mathrm{r}}^{\mathrm{p}}(T) = J_{\mathrm{p}} \pi_{\mathrm{l}}^{\mathrm{p}}(T) J_{\mathrm{p}}$. The state $\omega_\beta^{\mathrm{p}}$ goes into a vector state with respect to the GNS vector

$$\Omega_\beta^{\mathrm{p}} = \sum_{j=1}^{\nu} \frac{\mathrm{e}^{-\beta E_j/2}}{\sqrt{\mathrm{Tr}(\mathrm{e}^{-\beta H})}} \, e_j \otimes e_j.$$

That is

$$\omega_\beta^{\mathrm{p}}(T) = \langle \Omega_\beta^{\mathrm{p}}, \pi_{\mathrm{l}}^{\mathrm{p}}(T) \Omega_\beta^{\mathrm{p}} \rangle.$$

Associated with the representation is the standard self-dual cone

$$\mathcal{P}_\beta^{\mathrm{p}} = \left\{ \pi_{\mathrm{l}}^{\mathrm{p}}(T) \pi_{\mathrm{r}}^{\mathrm{p}}(T) \Omega_\beta^{\mathrm{p}} \,\middle|\, T \in \mathfrak{M}_{\mathrm{p}} \right\} = \left\{ T \otimes \overline{T} \, \Omega_\beta^{\mathrm{p}} \,\middle|\, T \in \mathfrak{M}_{\mathrm{p}} \right\},$$

which is invariant under J_{p}.

There are a priori many ways to lift the dynamics on the algebra $\mathfrak{M}_{\mathrm{p}}$ to a unitarily implemented dynamics on its image $\pi_{\mathrm{l}}^{\mathrm{p}}(\mathfrak{M}_{\mathrm{p}})$ inside $\mathcal{B}(\mathcal{K} \otimes \mathcal{K})$. The simplest choice is to use $K \otimes \mathbb{1}_{\mathcal{K}}$ as generator. However, there are two other natural choices available to us, one is the $\Omega_\beta^{\mathrm{p}}$-Liouvillean, fixed by requiring

$$L\Omega_\beta^{\mathrm{p}} = 0. \tag{1.2}$$

The other being the standard Liouvillean, selected uniquely by the requirement

$$\mathrm{e}^{\mathrm{i}tL}\mathcal{P}_\beta^{\mathrm{p}} \subseteq \mathcal{P}_\beta^{\mathrm{p}}. \tag{1.3}$$

For faithful states – as is the case here – the two Liouvilleans coincide and we get:

$$L_{\mathrm{p}} = K \otimes \mathbb{1}_{\mathcal{K}} - \mathbb{1}_{\mathcal{K}} \otimes K. \tag{1.4}$$

This is the unique choice satisfying either (1.2) or (1.3), and unitarily implementing the dynamics

$$\pi_{\mathrm{l}}^{\mathrm{p}}(\mathrm{e}^{\mathrm{i}tK} T \mathrm{e}^{-\mathrm{i}tK}) = \mathrm{e}^{\mathrm{i}tL_{\mathrm{p}}} \pi_{\mathrm{l}}^{\mathrm{p}}(T) \mathrm{e}^{-\mathrm{i}tL_{\mathrm{p}}}.$$

The GNS vector $\Omega_\beta^{\mathrm{p}}$ is a β-KMS vector for the dynamics $\tau_{\mathrm{p}}^t(T) = \mathrm{e}^{\mathrm{i}tL_{\mathrm{p}}} T \mathrm{e}^{-\mathrm{i}tL_{\mathrm{p}}}$ on $\pi_{\mathrm{l}}^{\mathrm{p}}(\mathfrak{M})$. Note that

$$\sigma(L_{\mathrm{p}}) = \sigma_{\mathrm{pp}}(L_{\mathrm{p}}) = \sigma(K) - \sigma(K) = \left\{ E_i - E_j \,\middle|\, 1 \le i, j \le n \right\},$$

with the obvious degeneracies. In particular $0 \in \sigma(L_{\mathrm{p}})$ is at least ν-fold degenerate.

1.2 The Reservoir

We begin by introducing the Fock representation of Weyl operators on the bosonic Fock space $\mathcal{F} = \Gamma(\mathfrak{h})$, build over the one-particle space $\mathfrak{h} = L^2(\mathbb{R}^3)$. That is,

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} \mathcal{F}^{(n)}, \quad \mathcal{F}^{(n)} = \mathfrak{h}^{\otimes_s n},$$

with the convention that $\mathfrak{h}^{\otimes_s 0} = \mathbb{C}$. Here $\otimes_s n$ denotes the n-fold symmetric tensor product, such that $\mathcal{F}^{(n)} = L^2_{\text{sym}}(\mathbb{R}^{3n})$, square integrable functions symmetric under interchange of the n variables.

We recall that the Segal fields $\phi(f)$, for $f \in \mathfrak{h}$, are given by

$$\phi(f) = \frac{1}{\sqrt{2}}(a^*(f) + a(f)),$$

where $a(f)$ and $a^*(f)$ are the bosonic annihilation and creation operators.

The Weyl operators in the Fock representation are given by the expression

$$W(f) = e^{\imath \phi(f)},$$

where f runs over the one-particle space $\mathfrak{h}$. We define $\mathfrak{W}$ to be the complex $*$-algebra generated by the Weyl operators, pertaining to functions f from.

$$\mathfrak{h}_0 = \left\{ f \in \mathfrak{h} \,\middle|\, |k|^{-\frac{1}{2}} f \in \mathfrak{h} \right\}. \tag{1.5}$$

Observe that this amounts to taking the linear span of the Weyl operators:

$$\mathfrak{W} = \text{span}\left\{ W(f) \,\middle|\, f \in \mathfrak{h}_0 \right\}. \tag{1.6}$$

That this yields the correct $*$-algebra follows from the Weyl relations

$$W(f)^* = W(-f) \quad \text{and} \quad W(f)W(g) = e^{\imath \text{Im}\langle f,g \rangle} W(f+g). \tag{1.7}$$

We remark that one should really pass on to the norm closure to get a C^*-algebra, but since we will eventually pass on to double commutants, i.e. von Neumann envelopes, we bypass the C^*-setting. This simplifies the construction of states and representations below. We will however still discuss the GNS representation of $\mathfrak{W}$ as if it was a C^*-algebra. See [48, 57] for how to extend states and representations to the intermediate C^*-algebra setting.

As a dynamics on the Weyl algebra, we take in these notes the second quantized massless dispersion relation $\mathbb{R}^3 \ni k \to |k|$. That is, we take as generator the self-adjoint non-negative operator

$$H_{\text{ph}} = d\Gamma(|k|),$$

with a single (up to a constant multiple) non-degenerate bound state $|0\rangle = (1, 0, 0, \dots)$, the vacuum vector. This is in fact the ground state of the reservoir Hamiltonian. That the Weyl algebra is invariant under the Heisenberg dynamics follows from the computation

$$e^{\imath t H_{\text{ph}}} W(f) e^{-\imath t H_{\text{ph}}} = W(e^{\imath t |k|} f), \tag{1.8}$$

together with linearity, cf. (1.6).

The associated β-KMS states are defined on Weyl operators, and extended by linearity to $\mathfrak{W}$, by the relation

$$\omega_\beta^{\mathcal{R}}(W(f)) = \mathrm{e}^{-\|\sqrt{1+2\rho_\beta}\,f\|^2/4}, \qquad (1.9)$$

where

$$\rho_\beta(k) = \frac{1}{\mathrm{e}^{\beta|k|} - 1} \qquad (1.10)$$

is Planck's thermal density for black body radiation. Note that $\rho_\beta(k) \sim (\beta|k|)^{-1}$ at $k = 0$, which is the reason for considering only Weyl operators for $f \in \mathfrak{h}_0$, cf. (1.5). It follows from the computation (1.8) that $\omega_\beta^{\mathcal{R}}$ are invariant states. The superscript $\mathcal{R}$ indicates that the state acts on the reservoir. The zero temperature state is the vector state

$$\omega_\infty^{\mathcal{R}}(W(f)) = \langle 0|W(f)|0\rangle = \mathrm{e}^{-\|f\|/4}. \qquad (1.11)$$

To construct the GNS representation of the Weyl algebra, with respect to the β-KMS state $\omega_\beta^{\mathcal{R}}$, we introduce so-called left (and right) Araki-Woods fields associated with the thermal density ρ_β from (1.10). The left fields are used to construct the GNS representation, whereas the right fields are kept for later use when we identify the standard form of the GNS representation.

The left and right Araki-Woods annihilation and creation operators are acting in $\mathcal{F} \otimes \mathcal{F}$, and defined for $f \in \mathfrak{h}_0$ by

$$\begin{aligned}
a_{\beta,\mathrm{l}}^{\mathrm{AW}}(f) &= a_\mathrm{l}(\sqrt{1+\rho_\beta}\,f) + a_\mathrm{r}^*(\sqrt{\rho_\beta}\,\bar{f}) \\
a_{\beta,\mathrm{r}}^{\mathrm{AW}}(f) &= a_\mathrm{r}(\sqrt{1+\rho_\beta}\,\bar{f}) + a_\mathrm{l}^*(\sqrt{\rho_\beta}\,f).
\end{aligned} \qquad (1.12)$$

Here, for $g \in \mathfrak{h}$,

$$a_\mathrm{l}(g) = a(g) \otimes \mathbb{1}_\mathcal{F}, \quad a_\mathrm{r}(g) = \mathbb{1}_\mathcal{F} \otimes a(g), \qquad (1.13)$$

and likewise for their adjoints $a_\mathrm{l}^*(g)$ and $a_\mathrm{r}^*(g)$. One can think of $g \to a_\mathrm{l}^\#(g)$ and $g \to a_\mathrm{r}^\#(\bar{g})$ as left and right zero-temperature Araki-Woods annihilation and creation operators. Here $a^\#$ denotes either a or a^*. The Araki-Woods operators form two (equivalent up to complex conjugation) non-Fock representations of the canonical commutation relations. They give rise to smeared Araki-Woods fields $\phi_{\beta,\mathrm{l}}^{\mathrm{AW}}(f)$ and $\phi_{\beta,\mathrm{r}}^{\mathrm{AW}}(f)$ and hence Weyl operators $W_{\beta,\mathrm{l}}^{\mathrm{AW}}(f)$ and $W_{\beta,\mathrm{r}}^{\mathrm{AW}}(f)$. We denote by

$$\widetilde{\mathfrak{W}}_{\beta,\mathrm{l}} = \mathrm{span}\left\{W_{\beta,\mathrm{l}}^{\mathrm{AW}}(f) \,\big|\, f \in \mathfrak{h}_0\right\} \quad \text{and} \quad \widetilde{\mathfrak{W}}_{\beta,\mathrm{r}} = \mathrm{span}\left\{W_{\beta,\mathrm{r}}^{\mathrm{AW}}(f) \,\big|\, f \in \mathfrak{h}_0\right\},$$

the left and right Araki-Woods Weyl algebras, as complex $*$-subalgebras of $\mathcal{B}(\mathcal{F} \otimes \mathcal{F})$.

The left and right Araki-Woods algebras form representations of the Weyl algebra by the prescriptions

$$\pi_{\beta,\mathrm{l}}^{\mathrm{AW}}(W(f)) = W_{\beta,\mathrm{l}}^{\mathrm{AW}}(f) \quad \text{and} \quad \pi_{\beta,\mathrm{r}}^{\mathrm{AW}}(W(f)) = W_{\beta,\mathrm{r}}^{\mathrm{AW}}(f),$$

and extended by linearity to $\mathfrak{W}$. The β-KMS state $\omega_\beta^{\mathcal{R}}$, cf. (1.9), goes into the β-independent vector state

$$\forall W \in \mathfrak{W}: \quad \omega_\beta^{\mathcal{R}}(W) = \langle 0 \otimes 0|\pi_{\beta,\mathrm{l}}^{\mathrm{AW}}(W)|0 \otimes 0\rangle$$

associated with the GNS vector $|0 \otimes 0\rangle := |0\rangle \otimes |0\rangle$. This follows from the observation that

$$\begin{aligned}
W_{\beta,\mathrm{l}}^{\mathrm{AW}}(f) &= W(\sqrt{1+\rho_\beta}f) \otimes W(\sqrt{\rho_\beta}\bar{f}) \\
W_{\beta,\mathrm{r}}^{\mathrm{AW}}(f) &= W(\sqrt{\rho_\beta}\bar{f}) \otimes W(\sqrt{1+\rho_\beta}f)
\end{aligned}$$

together with (1.9) and (1.11). Note that it follows from the Weyl relation (1.7) that the left and right Araki-Woods Weyl operators (at the same inverse temperature) commute, i.e.

$$\widetilde{\mathfrak{W}}_{\beta,\mathrm{l}} \subseteq \left(\widetilde{\mathfrak{W}}_{\beta,\mathrm{r}}\right)' \quad \text{and} \quad \widetilde{\mathfrak{W}}_{\beta,\mathrm{r}} \subseteq \left(\widetilde{\mathfrak{W}}_{\beta,\mathrm{l}}\right)'. \tag{1.14}$$

The dynamics on the Weyl-algebra in the GNS representation can again be implemented in several ways by a unitary group generated by a self-adjoint operator $L_{\mathcal{R}}$. The requirement

$$L_{\mathcal{R}} |0 \otimes 0\rangle = 0$$

selects the unique generator

$$L_{\mathcal{R}} = H_{\mathrm{ph}} \otimes \mathbb{1}_{\mathcal{F}} - \mathbb{1}_{\mathcal{F}} \otimes H_{\mathrm{ph}},$$

which we call the reservoir $|0 \otimes 0\rangle$-Liouvillean. It unitarily implements the dynamics in the GNS representation

$$\pi_{\beta,\mathrm{l}}^{\mathrm{AW}}(\mathrm{e}^{\mathrm{i}tH_{\mathrm{ph}}} A \mathrm{e}^{-\mathrm{i}tH_{\mathrm{ph}}}) = \mathrm{e}^{\mathrm{i}tL_{\mathcal{R}}} \pi_{\beta,\mathrm{l}}^{\mathrm{AW}}(A) \mathrm{e}^{-\mathrm{i}tL_{\mathcal{R}}}.$$

The GNS vector $|0 \otimes 0\rangle$ is a β-KMS vector for the dynamics on $\widetilde{\mathfrak{W}}_{\beta,\mathrm{l}}$. Note that $\sigma(L_{\mathcal{R}}) = \mathbb{R}$ is purely absolutely continuous, except for a non-degenerate eigenvalue at 0, with the β-KMS vector $|0 \otimes 0\rangle$ as eigenvector.

When we later start to perturb the free Pauli-Fierz dynamics, we will be forced to leave the complex algebra – as well as the C^*-algebra – setting and pass on to the enveloping Araki-Woods von Neumann algebras

$$\mathfrak{W}_{\beta,\mathrm{l}} = \left(\widetilde{\mathfrak{W}}_{\beta,\mathrm{l}}\right)'' \quad \text{and} \quad \mathfrak{W}_{\beta,\mathrm{r}} = \left(\widetilde{\mathfrak{W}}_{\beta,\mathrm{r}}\right)''.$$

The GNS vector $|0 \otimes 0\rangle$ becomes a cyclic faithful vector for the left algebra $\mathfrak{W}_{\beta,\mathrm{l}}$, and hence the algebra is in its standard representation. That the extension is faithful follows from $|0 \otimes 0\rangle$ being cyclic for the commutant, cf. (1.14).

The modular conjugation on the reservoir $J_{\mathcal{R}}$ is simply given by

$$J_{\mathcal{R}}(\varphi \otimes \psi) = \Gamma(\mathbf{c})\psi \otimes \Gamma(\mathbf{c})\varphi,$$

and extended to $\mathcal{F} \otimes \mathcal{F}$ by linearity and continuity. Here $\Gamma(\mathbf{c})$ is the lifting of complex conjugation $\mathbf{c}(f) = \bar{f}$ on $\mathfrak{h}$ to $\mathcal{F}$ using Segal's second quantization functor Γ. The modular conjugation intertwines the left and right representations

$$\forall A \in \mathfrak{W}: \qquad \pi_{\beta,\mathrm{r}}^{\mathrm{AW}}(A) = J_{\mathcal{R}} \pi_{\beta,\mathrm{l}}^{\mathrm{AW}}(A) J_{\mathcal{R}}$$

and we can identify the right algebra with the commutant of the left algebra and vice versa:

$$\left(\mathfrak{W}_{\beta,\mathrm{l}}\right)' = J_{\mathcal{R}} \mathfrak{W}_{\beta,\mathrm{l}} J_{\mathcal{R}} = \mathfrak{W}_{\beta,\mathrm{r}}.$$

The dynamics $W \to \tau_{\mathcal{R}}^t(W) = \mathrm{e}^{\mathrm{i}tL_{\mathcal{R}}} W \mathrm{e}^{-\mathrm{i}tL_{\mathcal{R}}}$ extends by continuity from $\widetilde{\mathfrak{W}}_{\beta,\mathrm{l}}$ to the von Neumann envelope $\mathfrak{W}_{\beta,\mathrm{l}}$, and – being unitarily implemented – is a σ-weakly continuous group of automorphisms. The GNS vector $|0 \otimes 0\rangle$ remains a β-KMS vector also for the extended dynamics on $\mathfrak{W}_{\beta,\mathrm{l}}$.

The extended dynamics on $\mathfrak{W}_{\beta,\mathrm{l}}$ is now (also) unitarily implemented by the standard Liouvillean, which due to faithfulness of the GNS state $|0 \otimes 0\rangle$ is identical to the

$|0\otimes 0\rangle$-Liouvillean $L_\mathcal{R}$. Recall that the standard Liouvillean is fixed by the requirement that it keeps the self-dual ($J_\mathcal{R}$-invariant) standard cone $\mathcal{P}_\beta^\mathcal{R}$ invariant, where

$$\mathcal{P}_\beta^\mathcal{R} = \overline{\left\{ BJ_\mathcal{R}B|0\otimes 0\rangle \,\middle|\, B \in \mathfrak{W}_{\beta,1} \right\}}.$$

We remark that the reader should think of the invariant state $|0\otimes 0\rangle$ as a background thermal photon cloud, with momentum distribution given by Planck's law (1.10). The left Fock space accounts for photons, and the right Fock space are holes. The left creation operators $a_{\beta,1}^{\mathrm{AW}*}$ add photons to the thermal background, while the left annihilation operators $a_{\beta,1}^{\mathrm{AW}}$ add holes. The vacuum state – the thermal background – is not annihilated by any Araki-Woods annihilation or creation operator. In this picture, the standard Liouvillean $L_\mathcal{R}$ can be interpreted as a Hamiltonian. Adding electrons cost energy, while adding holes release energy.

1.3 Non-interacting Pauli-Fierz Systems

Tensoring the small quantum system with the reservoir yields the non-interacting Pauli-Fierz system. The Hilbert space for the full system is

$$\mathcal{H} = \mathcal{K} \otimes \mathcal{F}$$

and the free Hamiltonian is

$$H_0 = K \otimes \mathbb{1}_\mathcal{F} + \mathbb{1}_\mathcal{K} \otimes H_{\mathrm{ph}}.$$

As the algebra of observables we take the (algebraic) tensor product

$$\mathfrak{M} = \mathfrak{M}_{\mathrm{p}} \otimes \mathfrak{W} \subseteq \mathcal{B}(\mathcal{H}).$$

The free Heisenberg dynamics generated by H_0 clearly preserves $\mathfrak{M}$.

Note that $\sigma(H_0) = [E_1, \infty)$ and $\sigma_{\mathrm{pp}}(H_0) = \sigma(K)$, hence; H_0's eigenvalues are all embedded in a half-axis of continuous spectrum. The ground state energy is E_1 with eigenvector $e_1 \otimes |0\rangle$.

We construct β-KMS states on $\mathfrak{M}$ be tensoring the relevant states on the constituent systems:

$$\omega_\beta(T \otimes W) = \omega_\beta^{\mathrm{p}}(T)\omega_\beta^\mathcal{R}(W),$$

where $T \in \mathfrak{M}_{\mathrm{p}} = \mathcal{B}(\mathcal{K})$ and $W \in \mathfrak{W}$.

We proceed to construct left and right representations at positive temperature by tensoring the two individual representations. Again, the left representation is the GNS representation, whereas the right is for extending the GNS representation to standard form at a later stage. We want representations on the Hilbert space

$$\mathcal{H}^{\mathrm{L}} = \mathcal{H} \otimes \mathcal{H} = \mathcal{K} \otimes \mathcal{F} \otimes \mathcal{K} \otimes \mathcal{F},$$

but obviously it is often more natural to construct operators on

$$\mathcal{H}_{\mathbf{s}}^{\mathrm{L}} = \mathcal{K} \otimes \mathcal{K} \otimes \mathcal{F} \otimes \mathcal{F}.$$

For this purpose we introduce a unitary transformation shuffling the tensor components:

$$\mathcal{H}^{\mathrm{L}} \ni u \otimes \varphi \otimes v \otimes \psi \to \mathbf{s}(u \otimes \varphi \otimes v \otimes \psi) = u \otimes v \otimes \varphi \otimes \psi \in \mathcal{H}_{\mathbf{s}}^{\mathrm{L}}, \qquad (1.15)$$

and extended by linearity and continuity.

We define
$$\pi_{\beta,1}^{\mathrm{PF}}(T \otimes W(f)) = \mathbf{s}^* \big(T \otimes \mathbb{1}_{\mathcal{K}} \otimes W_{\beta,1/\mathrm{r}}^{\mathrm{AW}}(f) \big) \mathbf{s},$$
$$\pi_{\beta,\mathrm{r}}^{\mathrm{PF}}(T \otimes W(f)) = \mathbf{s}^* \big(\mathbb{1}_{\mathcal{K}} \otimes \overline{T} \otimes W_{\beta,\mathrm{r}}^{\mathrm{AW}}(f) \big) \mathbf{s},$$
$$(1.16)$$

and extend by linearity to $\mathfrak{M}$. Write
$$\widetilde{\mathfrak{M}}_{\beta,1} = \pi_{\beta,1}^{\mathrm{PF}}(\mathfrak{M}) \quad \text{and} \quad \widetilde{\mathfrak{M}}_{\beta,\mathrm{r}} = \pi_{\beta,\mathrm{r}}^{\mathrm{PF}}(\mathfrak{M})$$

for the image algebras which are complex $*$-subalgebras of $\mathcal{B}(\mathcal{H}^{\mathrm{L}})$.

The β-KMS state ω_β is represented by the GNS vector state
$$\omega_\beta(A) = \langle \Omega_\beta^{\mathrm{PF}}, \pi_{\beta,1}^{\mathrm{PF}}(A) \Omega_\beta^{\mathrm{PF}} \rangle,$$

where the GNS vector is
$$\Omega_\beta^{\mathrm{PF}} = \mathbf{s}^* \big(\Omega_\beta^{\mathrm{p}} \otimes |0 \otimes 0\rangle \big).$$

Finally we can single out the free $\Omega_\beta^{\mathrm{PF}}$-Liouvillean as the sum of the particle and reservoir Liouvilleans:
$$L_0 = H_0 \otimes \mathbb{1}_{\mathcal{H}} - \mathbb{1}_{\mathcal{H}} \otimes H_0 = \mathbf{s}^* \big(L_{\mathrm{p}} \otimes \mathbb{1}_{\mathcal{F} \otimes \mathcal{F}} + \mathbb{1}_{\mathcal{K} \otimes \mathcal{K}} \otimes L_{\mathcal{R}} \big) \mathbf{s}.$$

Again the property $L_0 \Omega_\beta^{\mathrm{PF}} = 0$ determines the choice of generator uniquely.

As for the spectrum of L_0, it is the sum of the spectra of L_{p} and $L_{\mathcal{R}}$, that is
$$\sigma(L_0) = \mathbb{R} \quad \text{and} \quad \sigma_{\mathrm{pp}}(L_0) = \sigma(L_{\mathrm{p}}) = \sigma(K) - \sigma(K).$$

In particular, 0 is an at least ν-fold degenerate eigenvalue.

We end this subsection by passing on to the von Neumann envelopes
$$\mathfrak{M}_{\beta,1} = \big(\widetilde{\mathfrak{M}}_{\beta,1} \big)'' \quad \text{and} \quad \mathfrak{M}_{\beta,\mathrm{r}} = \big(\widetilde{\mathfrak{M}}_{\beta,\mathrm{r}} \big)''.$$

As for the reservoir, $\Omega_\beta^{\mathrm{PF}}$ is also a cyclic and faithful state on $\mathfrak{M}_{\beta,1}$ and hence; $\mathfrak{M}_{\beta,1}$ is in its standard representation. We identify the modular conjugation to be
$$J = \mathbf{s}^* \big(J_{\mathrm{p}} \otimes J_{\mathcal{R}} \big) \mathbf{s}, \tag{1.17}$$

intertwining the left and right algebras
$$\big(\mathfrak{M}_{\beta,1} \big)' = J \mathfrak{M}_{\beta,1} J = \mathfrak{M}_{\beta,\mathrm{r}}.$$

As for the reservoir, the dynamics $A \to \tau_0^t(W) = \mathrm{e}^{\mathrm{i}t L_0} W \mathrm{e}^{-\mathrm{i}t L_0}$ extends by continuity from $\widetilde{\mathfrak{M}}_{\beta,1}$ to the von Neumann envelope $\mathfrak{M}_{\beta,1}$, and – being unitarily implemented – is a σ-weakly continuous group of automorphisms. The GNS vector $\Omega_\beta^{\mathrm{PF}}$ remains a β-KMS vector also for the extended dynamics on the von Neumann algebra $\mathfrak{M}_{\beta,1}$.

The extended dynamics is (also) unitarily implemented by the standard Liouvillean, which is fixed uniquely by the demand that $\mathrm{e}^{\mathrm{i}t L}$ preserves the standard cone
$$\mathcal{P}_\beta^{\mathrm{PF}} = \overline{\big\{ A J A \Omega_\beta^{\mathrm{PF}} \,\big|\, A \in \mathfrak{M}_{\beta,1} \big\}}. \tag{1.18}$$

Again, the standard Liouvillean coincides with the $\Omega_\beta^{\mathrm{PF}}$-Liouvillean L_0.

Note the intertwining property $L_0 J = -J L_0$, which is consistent with the symmetric structure of the spectrum of L_0.

1.4 Interacting Pauli-Fierz Systems

In order to obtain an interacting system we add a perturbation to the free Hamiltonian, which couples the particle system and the reservoir. In this contribution we consider couplings linear in the field operators.

The object carrying the coupling is a function $G \in L^2(\mathbb{R}^3; \mathcal{B}(\mathcal{K}))$, an $M_\nu(\mathbb{C})$-valued square integrable function. The perturbation is then of the form

$$\phi^{\mathrm{PF}}(G) := \frac{1}{\sqrt{2}} \int_{\mathbb{R}^3} \left\{ G(k) \otimes a^*(k) + G(k)^* \otimes a(k) \right\} \mathrm{d}k,$$

acting in $\mathcal{H} = \mathcal{K} \otimes \mathcal{F}$. This can be thought of as a Segal field and indeed; if $\nu = 1$, it is a Segal field.

The interacting Pauli-Fierz Hamiltonian is given by

$$H = H_0 + \phi^{\mathrm{PF}}(G) = K \otimes \mathbb{1}_{\mathcal{F}} + \mathbb{1}_{\mathcal{K}} \otimes H_{\mathrm{ph}} + \phi^{\mathrm{PF}}(G).$$

Discussions of self-adjointness is postponed to the next chapter. We call H the *Pauli-Fierz Hamiltonian*.

At this point we meet a fundamental issue arising when studying interacting dynamics on Weyl-algebras. The Heisenberg evolution is no longer going to preserve the algebra of observables $\mathfrak{M}$. By the Trotter product formula [58, Thm. VIII.31] one sees that the double commutant $\mathfrak{M}''$ (the closure of $\mathfrak{M}$ in the weak operator topology) is invariant, but since the Weyl-algebra acts irreducibly on Fock space we find that $\mathfrak{M}'' = \mathcal{B}(\mathcal{H})$.

On the positive temperature side, we can perturb the free dynamics τ_0^t on the algebra $\mathfrak{M}_{\beta,\mathrm{l}}$. Being a σ-weakly continuous group of automorphisms on $\mathfrak{M}_{\beta,\mathrm{l}}$ it has a generator δ which is a closed operator on $\mathfrak{M}_{\beta,\mathrm{l}}$ as a Banach space. Using Araki-Dyson expansions one can add to δ perturbations of the form $[A, \cdot]$, for self-adjoint $A \in \mathfrak{M}_{\beta,\mathrm{l}}$, and construct a perturbed dynamics with $\delta + [A, \cdot]$ as generator. Araki derived the form of the standard Liouvillean unitarily implementing the perturbed dynamics to be $L_0 + A - JAJ$.

Dereziński, Jakšić and Pillet [17] extended the analysis of Araki to self-adjoint operators affiliated with $\mathfrak{M}_{\beta,\mathrm{l}}$. The precise construction of the perturbation we consider can be found in Subsect. 2.2. Here we just heuristically explain its structure. Formally we take the zero temperature perturbation $\phi^{\mathrm{PF}}(G)$ in its positive temperature form "$\pi_{\beta,\mathrm{l}}^{\mathrm{PF}}(\phi^{\mathrm{PF}}(G))$". For bosons this is not meaningful, but for fermions where field operators are bounded this can be taken literally. One can try and circumvent this by constructing a strongly continuous one-parameter group of unitaries "$s \to \pi_{\beta,\mathrm{l}}^{\mathrm{PF}}\left(\mathrm{e}^{\mathrm{i}\phi^{\mathrm{PF}}(sG)} \right)$". In general however, the operators $\mathrm{e}^{\mathrm{i}t\phi^{\mathrm{PF}}(G))}$ may not be in the (norm closure) of $\mathfrak{M}$. An exception to this is the spin-boson model, or more generally G's of the form $G(k) = G_0 g(k)$, with G_0 a self-adjoint matrix.

For now we simply write $\phi_{\beta,\mathrm{l}}^{\mathrm{PF}}(G)$ for the formal positive temperature Pauli-Fierz field obtained by formally applying the left representation $\pi_{\beta,\mathrm{l}}^{\mathrm{PF}}$ to $\phi^{\mathrm{PF}}(G)$. In Subsect. 3.4 we argue that $\phi_{\beta,\mathrm{l}}^{\mathrm{PF}}(G)$ constructed this way is indeed affiliated with $\mathfrak{M}_{\beta,\mathrm{l}}$.

One can now argue that

$$L_\beta = L_0 + \phi_{\beta,\mathrm{l}}^{\mathrm{PF}}(G) - J\phi_{\beta,\mathrm{l}}^{\mathrm{PF}}(G)J$$

is essentially self-adjoint on the intersection of the domains of the three unbounded summands. This perturbed Liouvillean generates a σ-weakly continuous group of automorphisms on $\mathfrak{M}_{\beta,\mathrm{l}}$, and by design it is already its own standard Liouvillean. It is a

fact immediate from Trotter's product formula that $\mathrm{e}^{\mathrm{i}tL_\beta}$ preserves the standard cone (1.18). We call L_β the *standard Pauli-Fierz Liouvillean*.

We remark at this stage that it is a result of Dereziński, Jakšić and Pillet [17], that the interacting Pauli-Fierz dynamics $A \to \tau_G^t(A) = \mathrm{e}^{\mathrm{i}tL_\beta} A \mathrm{e}^{-\mathrm{i}tL_\beta}$ – under suitable assumptions – also admits a normalized faithful β-KMS vector $\Omega_{\beta,G}^{\mathrm{PF}}$. A result which goes back to Araki for bounded perturbations from $\mathfrak{M}_{\beta,1}$.

1.5 (Open) Problems I

In this subsection we would like to present some of the questions one is interested in regarding Pauli-Fierz systems at zero and positive temperature. Our focus is on fully coupled Pauli-Fierz systems, that is we largely ignore questions relevant primarily for the weak coupling regime, which is much better understood than the fully coupled regime. Some problems are open, some will be either resolved or discussed in the following sections.

Zero Temperature:

- Under what conditions on G does the Hamiltonian admit a ground state, that is; when is $\inf \sigma(H)$ an eigenvalue. When is it non-degenerate?

- What can one say about the general structure of point spectrum?

- What can one say about general regularity properties of eigenstates?

- Under what conditions does the excited states vanish due to energy being dispersed by the radiation field?

- Under what conditions is the underlying continuous spectrum absolutely continuous?

- Can one prove asymptotic completeness, i.e. show that the canonical wave operators $W_\pm$ are unitary?

Positive Temperature:

- Under what conditions does the interacting Pauli-Fierz dynamics τ_G^t on $\mathfrak{M}_{\beta,1}$ admit a β-KMS state?

- Under what conditions is the β-KMS state the only invariant normal state?

- Under what conditions is the interacting Pauli-Fierz dynamics *ergodic*:

$$\forall \Psi \in \mathcal{P}_\beta^{\mathrm{PF}}, A \in \mathfrak{M}_{\beta,1}: \quad \lim_{T \to \infty} \frac{1}{T} \int_0^\infty \langle \Psi, \tau_G^t(A)\Psi \rangle \, \mathrm{d}t = \langle \Omega_{\beta,G}^{\mathrm{PF}}, A\Omega_{\beta,G}^{\mathrm{PF}} \rangle.$$

  Recall that the normal states on $\mathfrak{M}_{\beta,1}$ are in one-one correspondence with vector states, pertaining to unit vectors from the standard cone $\mathcal{P}_\beta^{\mathrm{PF}}$.

- Under what conditions is the interacting Pauli-Fierz dynamics *mixing*:

$$\forall \Psi \in \mathcal{P}_\beta^{\mathrm{PF}}, A \in \mathfrak{M}_{\beta,1}: \quad \lim_{T \to \infty} \langle \Psi, \tau_G^t(A)\Psi \rangle = \langle \Omega_{\beta,G}^{\mathrm{PF}}, A\Omega_{\beta,G}^{\mathrm{PF}} \rangle.$$

  The mixing property is sometimes referred to as "Return to Equilibrium".

- Can one prove an asymptotic completeness property, i.e. if the β-KMS state is the only invariant state, is there a unitary "wave operator" intertwining the interacting Pauli-Fierz dynamical system and the Araki-Woods dynamical system for the reservoir.

We make some clarifying remarks pertaining to the questions above.

In order to ensure uniqueness of an existing interacting ground state for H, one is typically forced to assume a priori that K's ground state is non-degenerate, that is $E_1 < E_2$. Otherwise one must rely on the coupling to induce a splitting of the ground state energy, which is difficult to control away from weak coupling. The typical tool here for the Hamiltonian is Perron-Frobenius methods. We will discuss a possible mechanism to invoke similar arguments for the Liouvillean.

The key tool for studying the structure of embedded point spectrum, as is the case here for H and L_β, as well as prove absence of singular continuous spectrum, is Mourre estimates. The deepest results established here will rely exactly on this technique, which goes back to Eric Mourre [52]. Pauli-Fierz Hamiltonians and standard Liouvilleans are however too singular to permit an application of standard implementations of the Mourre method [4]. Instead we really on singular Mourre theory developed in [27, 51, 62].

Suppose one can establish that H has a non-degenerate ground state, no excited states, and purely absolutely continuous spectrum above the ground state. Then it is a direct consequence of the spectral theorem and the Riemann-Lebesgue lemma that

$$\forall \psi, \varphi \in \mathcal{H}: \quad \lim_{t \to \infty} \langle \varphi, e^{-\imath t (H - E_0)} \psi \rangle = \langle \varphi, \Omega_0 \rangle \langle \Omega_0, \psi \rangle,$$

where Ω_0 is a normalized ground state and E_0 the corresponding ground state energy. This is sometimes called "Approach to the Ground State" and is the zero-temperature analogue of mixing.

By regularity properties of bound states we refer here primarily to number bounds, which have different interpretations at zero and positive temperature. At zero temperature, bounds such as $\langle \psi, N^k \psi \rangle < \infty$ gives control over the infrared catastrophe, in that it controls the number of soft photons a bound state carries. Here $N = \mathbb{1}_{\mathcal{K}} \otimes d\Gamma(\mathbb{1}_{\mathfrak{h}})$ is the number operator and ψ is a bound state. At positive temperature one has both a photon counter $N \otimes \mathbb{1}_{\mathcal{H}}$ and a hole counter $\mathbb{1}_{\mathcal{H}} \otimes N$. Subtracting one from the other yields a total photon counter, a "charge" operator. A deformed thermal photon cloud may be very far from the Planck distribution while having finite (total) photon number. A better measure for controlling the photon/hole content of a bound state is the sum of photons *and* holes $N \otimes \mathbb{1}_{\mathcal{H}} + \mathbb{1}_{\mathcal{H}} \otimes N$, which is what we understand by the number operator at positive temperature. Other relevant regularity questions pertain to the study of momentum content, in particular in the infrared region, of bound states. This amounts to studying regularity of $k \to a(k)\psi$.

Asymptotic completeness at zero temperature expresses that states in the absolutely continuous subspace correspond exactly to scattering processes with incoming photons entering the interaction region from spatial infinity, exciting/relaxing the atom and escaping again to spatial infinity leaving the atom in an altered state. To express this more concisely set $\mathcal{V} = \mathcal{H}_{\mathrm{pp}}$, the closure of the linear span of all eigenstates of H, which may be a one-dimensional subspace of $\mathcal{H}$. The spaces of incoming and outgoing states are identical and equal $\mathcal{H}_\pm = \mathcal{V} \otimes \mathcal{F}$.

We define a scattering identification operator $I: \mathcal{H}_\pm \to \mathcal{H}$ by

$$I(\psi \otimes a^*(f)^n |0\rangle) = a^*(f)^n \psi,$$

and extended (formally) by linearity to $\mathcal{H}_\pm$. The wave operators as maps $W_\pm: \mathcal{H}_\pm \to \mathcal{H}$ are now given by

$$W_\pm = \mathrm{s} - \lim_{t \to \infty} e^{\imath t H} I e^{-\imath t H_\pm},$$

where $H_{\pm} = H_{|\mathcal{V}} \otimes \mathbb{1}_{\mathcal{F}} + \mathbb{1}_{\mathcal{V}} \otimes H_{\mathrm{ph}}$ is the free dynamics on the incoming and outgoing states. Asymptotic completeness amounts to establishing existence and unitarily of wave operators. The scattering operator $S = W_+^* W_- : \mathcal{H}_- \to \mathcal{H}_+$ is then a unitary map from incoming to outgoing states. Wave operators are known to exist (under reasonable assumptions), but unitarity has only very recently been established by Faupin and Sigal, for the particular case of the spin-boson model and in the weak coupling regime. The general problem remains open. We will not be discussing asymptotic completeness further, but the techniques we develop have in other contexts been essential to establishing asymptotic completeness. For further material we refer the reader to [13, 21, 22, 30].

Return to equilibrium, that is mixing, or its weaker form ergodicity expresses the following physical situation. When the Pauli-Fierz system is uncoupled, the atomic levels are not mixed, and hence the system is really a sum of one-dimensional systems. In particular, the reservoir can not drive the small system towards the equilibrium vector state given by $\Omega^P F_\beta$. However, if the coupling does mix the atomic energy levels, then one expects that any normal state on $\mathfrak{M}_{\beta,1}$ should evolve towards the interacting β-KMS vector state given by $\Omega_{\beta,G}^{\mathrm{PF}}$. Note that the final inverse temperature is enforced on the small system by the reservoir.

As for asymptotic completeness at positive temperature, this is a completely unexplored question. As far as the author is aware, there is not even a concise strategy proposed to address this problem.

We have now arrived at the starting point, substance wise, for the positive temperature side of these notes. The following quantum analogue of the classical Koopman theorem [58], expresses ergodicity and mixing in terms of spectral properties of generators. The translation of "Return to Equilibrium" into a spectral problem is often referred to as quantum Koopmanism.

Theorem 1.1. *We have the following at positive temperature $\beta \in (0, \infty)$:*

1. *The interacting Pauli-Fierz dynamics at positive temperature is ergodic if and only if $\sigma_{\mathrm{pp}}(L_\beta) = \{0\}$ and 0 is a simple eigenvalue. (The corresponding eigenvector being the β-KMS vector.)*

2. *The interacting Pauli-Fierz dynamics at positive temperature is mixing if and only if $\mathrm{w} - \lim_{t \to \infty} \mathrm{e}^{\mathrm{i}tL_\beta} = |\Omega_\beta^{\mathrm{PF}}\rangle\langle\Omega_\beta^{\mathrm{PF}}|$.*

3. *If the spectrum of L_β is purely absolutely continuous, except for a simple eigenvalue at 0, then the interacting Pauli-Fierz dynamics is mixing.*

We have now set the stage for embarking on a systematic spectral analysis of Pauli-Fierz Hamiltonians and standard Pauli-Fierz Liouvilleans.

2 Construction and Properties of Operators

In this section we construct the Pauli-Fierz Hamiltonian and its positive temperature counterpart, the standard Pauli-Fierz Liouvillean. Furthermore, we establish some of their basic properties. We will throughout these notes make heavy use of the $C^1(A)$ commutator calculus. For the convenience of the reader we have, in the form of Appendix A, included a condensed presentation of the elements of the calculus that we rely on.

2.1 The Pauli-Fierz Hamiltonian

Recall from Subsect. 1.1 that as a small quantum system we took a finite dimensional Hilbert space $\mathcal{H} = \mathbb{C}^\nu$ with Hamiltonian $K \in \mathrm{M}_\nu(\mathbb{C})$, a self-adjoint $\nu \times \nu$ matrix $K^* = K$. In fact we chose K to be diagonal with its real eigenvalues sitting on the diagonal.

The dispersion relation for the field is the massless relativistic relation $k \to |k|$ considered as a multiplication operator on $\mathfrak{h} = L^2(\mathbb{R}^3)$. This gives rise to the second quantized free field energy $H_{\mathrm{ph}} = \mathrm{d}\Gamma(|k|)$, as a self-adjoint operator on the bosonic Fock-space $\mathcal{F} = \Gamma(\mathfrak{h}) = \oplus_{\ell=0}^\infty \mathfrak{h}^{\otimes_s \ell}$. We write $|0\rangle = (1, 0, 0, \dots)$ for the vacuum state in $\mathcal{F}$. Our inner products will always be conjugate linear in the first variable, and linear in the second.

We define a class of admissible coupling operators/functions

$$G \in \mathcal{B}(\mathcal{K}; \mathcal{K} \otimes \mathfrak{h}) = L^2\big(\mathbb{R}^3; \mathrm{M}_\nu(\mathbb{C})\big).$$

That the two spaces above can be identified can be seen as follows: If $G \colon \mathbb{R}^3 \to \mathrm{M}_\nu(\mathbb{C})$ is square integrable one can define a bounded operator $B_G \in \mathcal{B}(\mathcal{K}; \mathcal{K} \otimes \mathfrak{h})$ by

$$(B_G v)(k) = G(k)v,$$

where we identified $\mathcal{K} \otimes \mathfrak{h}$ isometrically with $L^2(\mathbb{R}^3; \mathbb{C}^\nu)$. Then

$$\|B_G\|^2 = \sup_{|v| \leq 1} \|B_G v\|^2 = \sup_{|v| \leq 1} \int_{\mathbb{R}^3} |G(k)v|^2 \, \mathrm{d}k \leq \|G\|^2$$

and the linear map $G \to B_G$ is a contraction, but it is not an isometry. To see that it is surjective with a bounded inverse, let $B \in \mathcal{B}(\mathcal{K}; \mathcal{K} \otimes \mathfrak{h})$ and define the candidate for an inverse G by $G_{ij}(k) = \langle (Be_i)(k), e_j \rangle$, where $e_1, \dots, e_\nu$ is the standard basis for $\mathbb{C}^\nu$. Then

$$\sum_{j=1}^\nu \int_{\mathbb{R}^3} |G_{ij}(k)|^2 \mathrm{d}k = \int_{\mathbb{R}^3} |(Be_i)(k)|^2 \mathrm{d}k = \|Be_i\|_{\mathcal{K} \otimes \mathfrak{h}}^2 \leq \|B\|^2.$$

Hence

$$\|G\|^2 = \int_{\mathbb{R}^3} \|G(k)\|^2 \mathrm{d}k \leq \sum_{1 \leq i,j \leq \nu} \int_{\mathbb{R}^3} |G_{ij}(k)|^2 \mathrm{d}k \leq \nu \|B\|^2.$$

From now on we will identify couplings G with elements of $L^2(\mathbb{R}^3; M_\nu(\mathbb{C}))$, and norms of couplings will be L^2-norms. We remark that the identification of coupling

operators as $\mathcal{B}(\mathcal{K})$-valued functions above is particular to finite dimensional small systems, cf. [19, Remark 5.1]. Let $\mu > 0$ be arbitrary, but fixed. For the coupling G we assume the existence of a constant $C > 0$ such that

(HGn)
$$\forall k \in \mathbb{R}^3, |k| \leq 1, \quad \text{and} \quad |\alpha| \leq n : \qquad \|\partial_k^\alpha G(k)\| \leq C|k|^{n-\frac{3}{2}+\mu-|\alpha|+\frac{\delta_{\alpha,0}}{2}}$$
$$\forall k \in \mathbb{R}^3, |k| \geq 1, \quad \text{and} \quad |\alpha| \leq n : \qquad \|\partial_k^\alpha G(k)\| \leq C|k|^{-\frac{3}{2}-\mu}.$$

The derivatives are distributional derivatives. We will make use of the condition **(HGn)** on G with $n = 0, 1, 2$.

The above conditions reflect that $|k|^{|\alpha|-n}\partial_k^\alpha G$ is slightly better than square integrable near zero, and the $\partial_k^\alpha G$'s are slightly better than square integrable at infinity. For our commutator estimates in Sect. 4 it will not suffice to demand just square integrability. We remark that there is nothing special about three dimensions or the dispersion $|k|$. For some results we could deal with infinite dimensional small system $\mathcal{K}$, and more singular G's. The above special case however captures the essentials, and permits us to formulate simple - yet pertinent - conditions that can be used for all our results at zero temperature.

We now define the free and coupled Hamiltonians as

$$H_0 = K \otimes \mathbb{1}_{\mathcal{F}} + \mathbb{1}_{\mathcal{K}} \otimes H_{\mathrm{ph}} \quad \text{and} \quad H = H_0 + \phi^{\mathrm{PF}}(G),$$

where

$$\phi^{\mathrm{PF}}(G) = \frac{1}{\sqrt{2}} \int_{\mathbb{R}^3} \left\{ G(k)^* a(k) + G(k) a^*(k) \right\} \mathrm{d}k. \tag{2.19}$$

In the following we will drop the superscript PF to simplify notation. The reader should be able to tell from the argument when ϕ denotes a Segal field and when it is of the coupling type (2.19). We remark that H_0 is self-adjoint on $\mathcal{D}(H_0) = \mathcal{D}(\mathbb{1}_{\mathcal{K}} \otimes H_{\mathrm{ph}})$ and that

$$\mathcal{C} = \mathcal{K} \otimes \Gamma_{\mathrm{fin}}(C_0^\infty(\mathbb{R}^3)) \tag{2.20}$$

is a core for H_0. Furthermore, as for Segal fields, by Nelson's analytic vector theorem $\phi(G)$ is essentially self-adjoint on $\mathcal{C}$ as well. In fact Segal fields are a special case, corresponding to $\nu = 1$. See [48] for a proof. The notation $\Gamma_{\mathrm{fin}}(V)$, with $V \subset \mathfrak{h}$ a subspace, denotes the algebraic direct sum of $V^{\otimes_s n}$, with tensor products of subspaces of Hilbert spaces always being algebraic, whereas tensor products of Hilbert spaces always denote Hilbert space tensor products, i.e. completion of algebraic tensor products.

We will as usual use the notation N for the number operator $\mathrm{d}\Gamma(\mathbb{1}_{\mathfrak{h}})$ as an operator on $\mathcal{F}$, and we will recycle the same notation on $\mathcal{H}$ instead of the more cumbersome $\mathbb{1}_{\mathcal{K}} \otimes N$.

Note the easy to verify bounds for field operators

Lemma 2.1. *Let* $G \in L^2(\mathbb{R}^3; \mathrm{M}_\nu(\mathbb{C}))$ *and* $\psi \in \mathcal{C}$.

1. The following bounds hold true for all $\sigma > 0$

$$\|\phi(G)\psi\| \leq \sqrt{2}\|G\|\|\sqrt{N}\psi\| + \frac{1}{\sqrt{2}}\|G\|\|\psi\| \tag{2.21}$$

$$|\langle \psi, \phi(G)\psi \rangle| \leq \sigma\langle \psi, N\psi \rangle + (2\sigma)^{-1}\|G\|^2\|\psi\|^2. \tag{2.22}$$

2. *If furthermore $|k|^{-1/2}G \in L^2(\mathbb{R}^3; \mathrm{M}_\nu(\mathbb{C}))$, then for all $\sigma > 0$*

$$\|\phi(G)\psi\| \leq \sqrt{2}\||k|^{-\frac{1}{2}}G\|\|(\mathbb{1}_{\mathcal{K}} \otimes \sqrt{H_{\mathrm{ph}}})\psi\| + \frac{1}{\sqrt{2}}\|G\|\|\psi\| \qquad (2.23)$$

$$|\langle \psi, \phi(G)\psi \rangle| \leq \sigma\langle \psi, \mathbb{1}_{\mathcal{K}} \otimes H_{\mathrm{ph}}\psi \rangle + (2\sigma)^{-1}\|G/\sqrt{|k|}\|^2\|\psi\|^2. \qquad (2.24)$$

The bounds in 1 extend by continuity to $\psi \in \mathcal{D}(\sqrt{N})$, whereas the bounds in 2 extend by continuity to $\psi \in \mathcal{D}(\mathbb{1}_{\mathcal{K}} \otimes \sqrt{H_{\mathrm{ph}}})$. We note that the bound (2.21) implies that $\mathcal{D}(\sqrt{N}) \subseteq \mathcal{D}(\phi(G))$, just as for Segal fields.

By (2.23) and Kato-Rellich's theorem, [59, Thm. X.12], H is essentially self-adjoint on $\mathcal{C}$ and $\mathcal{D}(H) = \mathcal{D}(H_0)$. In particular, the domain of H does not depend on G.

The bound (2.24) in particular implies that H is bounded from below. We furthermore observe that if we equip the space of G's satisfying **(HG0)** with the norm $\|G\|_0^2 = \int_{\mathbb{R}^3}(1 + |k|^{-1})\|G(k)\|^2 \mathrm{d}k$, then the resolvent map $(z, G) \to (H - z)^{-1}$ is norm continuous. Here $\mathrm{Im}z \neq 0$. We introduce notation for the bottom of H's spectrum

$$\Sigma = \inf \sigma(H) > -\infty. \qquad (2.25)$$

The spectrum of H is in fact a half-line starting at Σ as we now proceed to prove, using an argument from [28]. We pass via two useful results on the way, the first being:

Lemma 2.2. *Assume* **(HG0)**. *Then H is of class $C^1_{\mathrm{Mo}}(N)$ and the operator representing the commutator form is $[H, N]^\circ = \imath\phi(\imath G)$.*

Proof. We aim to use Proposition A.9 to establish the lemma. Note first that the property 2a in Proposition A.9 2 trivially holds true since $\mathcal{D}(H) = \mathcal{D}(H_0)$ and N commutes with H_0.

Secondly, since $H_0 + N$ is essentially self-adjoint on $\mathcal{C}$ – being a direct sum of multiplication operators – we conclude that $\mathcal{C}$ is dense in $\mathcal{D}(H) \cap \mathcal{D}(N) = \mathcal{D}(H_0 + N)$ with respect to the intersection topology.

We can now compute in the sense of forms on $\mathcal{C}$

$$[H, N] = [\phi(G), N] = \imath\phi(\imath G).$$

Since $\phi(\imath G)$ is $N^{1/2}$-bounded the above form identity now extends to the intersection domain $\mathcal{D}(H) \cap \mathcal{D}(N)$. By (2.23) (applied with G replaced by $\imath G$) we thus find that property 2b in Proposition A.9 2 is also satisfied and hence, H is of class $C^1_{\mathrm{Mo}}(N)$. $\qquad \square$

The second ingredient is a version of the so-called pull through formula

Proposition 2.3. *Suppose* **(HG0)**. *For any $z \in \mathbb{C}\backslash[\Sigma, \infty)$ and $\psi \in \mathcal{D}(\sqrt{N})$ we have as an $L^2(\mathbb{R}^3; \mathcal{H})$-identity*

$$a(k)(H - z)^{-1}\psi = (H + |k| - z)^{-1}a(k)\psi - \frac{1}{\sqrt{2}}(H + |k| - z)^{-1}(G(k) \otimes \mathbb{1}_{\mathcal{F}})\psi.$$

Remark 2.4. The fact that H is of class $C^1_{\mathrm{Mo}}(N)$, and hence in particular of class $C^1(N)$, implies that $\mathcal{D}(N)$, and by interpolation $\mathcal{D}(\sqrt{N})$, is preserved by resolvents of H. See Lemma A.5 2. Hence both sides of the pull through formula defines elements of $L^2(\mathbb{R}^3; \mathcal{H})$.

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Proof. Let $\tilde{\psi} \in \mathcal{C}$ and compute

$$a(k)(H - z)\tilde{\psi} = (H + |k| - z)a(k)\tilde{\psi} + \frac{1}{\sqrt{2}}(G(k) \otimes \mathbb{1}_{\mathcal{F}})\tilde{\psi}$$

as an $L^2(\mathbb{R}^3; \mathcal{H})$-identity, where the only possibly irregular contribution is G near zero. Since $z - |k| \in \rho(H)$ - the resolvent set for H - we obtain the $L^2(\mathbb{R}^3; \mathcal{H})$-identity

$$a(k)\tilde{\psi} = (H + |k| - z)^{-1}a(k)(H - z)\tilde{\psi} - \frac{1}{\sqrt{2}}(H + |k| - z)^{-1}(G(k) \otimes \mathbb{1}_{\mathcal{F}})\tilde{\psi}.$$

Let $h \in L^2(\mathbb{R}^3)$ and $\varphi \in \mathcal{C}$. Then

$$\langle a^*(h)\varphi, \tilde{\psi} \rangle = \langle \tilde{\varphi}, (H - z)\tilde{\psi} \rangle - \int_{\mathbb{R}^3} \frac{\overline{h(k)}}{\sqrt{2}} \langle \varphi, (H + |k| - z)^{-1}(G(k) \otimes \mathbb{1}_{\mathcal{F}})\tilde{\psi} \rangle \, dk,$$

where

$$\tilde{\varphi} = \int_{\mathbb{R}^3} h(k)a^*(k)(H + |k| - z)^{-1}\varphi \, dk \in \mathcal{H}.$$

From this expression, and H being essentially self-adjoint on $\mathcal{C}$, we observe that the above identity remains true for $\tilde{\psi} \in \mathcal{D}(H)$. Inserting $\tilde{\psi} = (H - z)^{-1}\psi$, where $\psi \in \mathcal{D}(\sqrt{N})$ yields the proposition. Here we used that $L^2(\mathbb{R}^3) \otimes \mathcal{C}$ (algebraic tensor product) is dense in $L^2(\mathbb{R}^3; \mathcal{H})$. $\square$

For stronger versions of the pull through formula see [12, 29]. We are now ready to show that the spectrum is a half-axis. The argument goes back to [28], cf. also [12].

Theorem 2.5. *Suppose* **(HG0)**. *Then* $\sigma(H) = [\Sigma, \infty)$.

Proof. It suffices to show that $\sigma((H - \Sigma + 1)^{-1}) \supset (0, 1]$. To see this, let $\lambda > 0, \epsilon > 0$ and choose $\tilde{\psi} \in \mathbb{1}[H \leq \Sigma + \epsilon/2]\mathcal{H}$ to be normalized. Since $\mathcal{C}$ is dense in $\mathcal{D}(H)$, we can pick a normalized $\psi \in \mathcal{C}$ such that $\|(H - \Sigma)(\tilde{\psi} - \psi)\| \leq \epsilon/2$ and hence we must have $\|(H - \Sigma)\psi\| \leq \epsilon$.

Choose a function $h \in C_0^\infty(\mathbb{R})$ real-valued with $\|h\| = 1$ and supp $h \subseteq [-1, 1]$. Put $h_n(k) = n^{3/2}h(n(|k| - \lambda))$. Form $\psi_n = a^*(h_n)\psi$ and compute for $\varphi \in \mathcal{D}(\sqrt{N})$ using the pull through formula Proposition 2.3

$$\langle \varphi, ((H - \Sigma + 1)^{-1} - (\lambda + 1)^{-1})\psi_n \rangle$$

$$= \int_{\mathbb{R}^3} h_n(k)\langle a(k)((H - \Sigma + 1)^{-1} - (\lambda + 1)^{-1})\varphi, \psi \rangle dk$$

$$= \int_{\mathbb{R}^3} h_n(k)\langle ((H + |k| - \Sigma + 1)^{-1} - (\lambda + 1)^{-1})a(k)\varphi, \psi \rangle dk$$

$$- \int_{\mathbb{R}^3} \frac{h_n(k)}{\sqrt{2}} \langle G(k) \otimes \mathbb{1}_{\mathcal{F}}\varphi, (H + |k| - \Sigma + 1)^{-1}\psi \rangle dk.$$

Since h_n goes to zero weakly in $L^2(\mathbb{R}^3)$, the last term is $o(1)\|\varphi\|$ in the limit of large n. To deal with the first term on the right-hand side we estimate using the support

properties of h_n and the choice of ψ:

$$\left| h_n(k)\left\langle \left((H + |k| - \Sigma + 1)^{-1} - (\lambda + 1)^{-1}\right)a(k)\varphi, \psi\right\rangle\right|$$

$$\leq \frac{|h_n(k)|}{\sqrt{|k|}}\left\|(H_{\mathrm{ph}} + |k| + 1)^{-1}\sqrt{|k|}a(k)\varphi\right\|$$

$$\times \left\|(H_{\mathrm{ph}} + |k| + 1)(H + |k| - \Sigma + 1)^{-1}\left((H - \Sigma) + (|k| - \lambda)\right)\psi\right\|$$

$$\leq C\left(\epsilon + \frac{1}{n}\right)\frac{|h_n(k)|}{\sqrt{|k|}}\left\|\sqrt{|k|}a(k)(H_{\mathrm{ph}} + 1)^{-1}\varphi\right\|.$$

Noting that $\|h_n/\sqrt{|k|}\| \leq (\lambda - 1/n)^{-1/2}\|h_n\| = (\lambda - 1/n)^{-1/2}$ we conclude from Cauchy-Schwartz that

$$\left|\left\langle \varphi, \left((H - \Sigma + 1)^{-1} - (\lambda + 1)^{-1}\right)\psi_n\right\rangle\right| \leq C(\epsilon + o(1))\|\varphi\|,$$

where $o(1)$ refers to the large n limit. It now remains to prove that $\|a^*(h_n)\psi\|$ is bounded away from zero. But this follows from the computation

$$\|a^*(h_n)\psi\|^2 = \|\psi\|^2 + \|a(h_n)\psi\|^2.$$

Recall that when h_n goes to zero weakly, we have $a(h_n)\psi \to 0$ in norm, whenever $\psi \in \mathcal{D}(\sqrt{N})$. □

We end this subsection introducing some extra structure that will be used in the next subsection.

We define a conjugate linear involution operator $\mathbf{C}$ on $\mathcal{H}$ as follows. It is a tensor product of two conjugate linear involutions, one on $\mathcal{K}$ and one on $\mathcal{F}$. On $\mathcal{K}$ we simply take coordinate wise complex conjugation $(\mathbf{c}v)_j = \bar{v}_j$, and on $\mathcal{F}$ we take second quantized complex conjugation $\Gamma(\mathbf{c})$, acting on an n-particle state by complex conjugation, or equivalently described by the intertwining $\Gamma(\mathbf{c})a^{\#}(g)\Gamma(\mathbf{c}) = a^{\#}(\mathbf{c}g)$. In conclusion $\mathbf{C} = \mathbf{c} \otimes \Gamma(\mathbf{c})$. Note that $\langle \mathbf{C}\psi, \varphi\rangle = \langle \mathbf{C}\varphi, \psi\rangle$.

With this choice of conjugation we can define $H^{\mathbf{c}} = \mathbf{C}H\mathbf{C} = H_0 + \phi(\overline{G})$. Note that $H_0^{\mathbf{c}} = \mathbf{C}H_0\mathbf{C} = H_0$. Clearly, the spectrum, pure point spectrum and absolutely/singular continuous spectrum of the two operators coincide. Eigenvectors are related by $\psi^{\mathbf{c}} = \mathbf{C}\psi$, where $H\psi = \lambda\psi$ and $H^{\mathbf{c}}\psi^{\mathbf{c}} = \lambda\psi^{\mathbf{c}}$. Finally we observe that the spectral resolutions E and $E^{\mathbf{c}}$ of the operators H and $H^{\mathbf{c}}$ are related by $E_{\psi}^{\mathbf{c}} = E_{\mathbf{C}\psi}$.

2.2 The Standard Pauli-Fierz Liouvillean

The Liouvillean, at inverse temperature $\beta > 0$, is a self-adjoint operator on the doubled Hilbert space $\mathcal{H}^{\mathrm{L}} := \mathcal{H} \otimes \mathcal{H}$. The zero temperature Liouvillean, corresponding to $\beta = \infty$, is given by

$$L_{\infty} = H \otimes \mathbb{1}_{\mathcal{H}} - \mathbb{1}_{\mathcal{H}} \otimes H^{\mathbf{c}},$$

which is essentially self-adjoint on algebraic tensor products $\mathcal{D} \otimes \mathcal{D}$, where $\mathcal{D} \subseteq \mathcal{H}$ is a core for H. See [58, Thm. VIII.33]. As a choice of core we take

$$\mathcal{C}^{\mathrm{L}} = \mathcal{C} \otimes \mathcal{C}, \tag{2.26}$$

where $\mathcal{C}$ was defined in (2.20). Observe that L_{∞} is unbounded from below and indeed $\sigma(L_{\infty}) = \mathbb{R}$.

We furthermore write

$$L_0 = H_0 \otimes \mathbb{1}_{\mathcal{H}} - \mathbb{1}_{\mathcal{H}} \otimes H_0$$

for the uncoupled Liouvillean. Recall that $H_0^{\mathsf{c}} = H_0$. With this notation, at least formally, the zero temperature ($\beta = \infty$) Liouvillean can be written as the operator sum $L_\infty = L_0 + \phi(G) \otimes \mathbb{1}_{\mathcal{H}} - \mathbb{1}_{\mathcal{H}} \otimes \phi(\overline{G})$.

We will need stronger conditions than **(HGn)** on the coupling G when dealing with the Liouvillean. Let $n \in \mathbb{N}_0$. We assume that G admits n distributional derivatives in $L^1_{\mathrm{loc}}(\mathbb{R}^3; M_\nu(\mathbb{C}))$ and the existence of a constant $C > 0$ such that

$$\textbf{(LGn)} \qquad \begin{aligned} &\forall k \in \mathbb{R}^3, |k| \le 1, \quad \text{and} \quad |\alpha| \le n: \qquad \|\partial_k^\alpha G(k)\| \le C|k|^{n-1+\mu-|\alpha|} \\ &\forall k \in \mathbb{R}^3, |k| \ge 1, \quad \text{and} \quad |\alpha| \le n: \qquad \|\partial_k^\alpha G(k)\| \le C|k|^{-\frac{3}{2}-\delta_{\alpha,0}-\mu}. \end{aligned}$$

We will make use of the condition **(LGn)** on G with $n = 0, 1, 2$. Note that **(LGn)** implies **(HGn)**. As for the Hamiltonian, there is nothing particular about dimension 3. The difference between **(HGn)** and **(LGn)** comes from having to absorb an infrared singularity from the Planck density (1.10), which mix the left (photons) and right (holes) field components at positive temperature, $\beta < \infty$. One could use a different density, modifying **(LGn)** accordingly. See also Remark 2.12.

We now give an explicit construction of the positive temperature perturbation denoted by $\phi_{\beta,\mathrm{l}}^{\mathrm{PF}}(G)$ and formally introduced in Subsect. 1.4.

For $G_{\mathrm{L}} \in L^2(\mathbb{R}^3; \mathcal{B}(\mathcal{K} \otimes \mathcal{K}))$ we extend the definition of the (zero temperature) left and right annihilation and creation operators to read

$$\begin{aligned} a_{\mathrm{l}}(G_{\mathrm{L}}) &= \mathbf{s}^* \Big(\int_{\mathbb{R}^3} G_{\mathrm{L}}(k)^* \otimes a(k) \otimes \mathbb{1}_{\mathcal{F}} \, dk \Big) \mathbf{s} \\ a_{\mathrm{r}}(G_{\mathrm{L}}) &= \mathbf{s}^* \Big(\int_{\mathbb{R}^3} G_{\mathrm{L}}(k)^* \otimes \mathbb{1}_{\mathcal{F}} \otimes a(k) \, dk \Big) \mathbf{s}. \end{aligned} \tag{2.27}$$

Here $\mathbf{s}$ is the unitary shuffle defined in (1.15). The creation operators $a_{\mathrm{l/r}}^*(G_{\mathrm{L}})$, adjoints of the annihilation operators, are represented by similar formulas.

For use as G_{L} we define

$$G_{\mathrm{l}}(k) = G(k) \otimes \mathbb{1}_{\mathcal{K}} \quad \text{and} \quad G_{\mathrm{r}}(k) = \mathbb{1}_{\mathcal{K}} \otimes \overline{G(k)}. \tag{2.28}$$

With this definition we have $J_{\mathrm{p}} G_{\mathrm{l}}(k) = G_{\mathrm{r}}(k) J_{\mathrm{p}}$, where J_{p} is the modular conjugation on the particle system (1.1).

Recalling the form of the Araki-Woods annihilation and creation operators (1.12), we can now define positive temperature left and right annihilation and creation operators

$$\begin{aligned} a_{\beta,\mathrm{l}}(G) &= a_{\mathrm{l}}(\sqrt{1+\rho_\beta}\, G_{\mathrm{l}}) + a_{\mathrm{r}}^*(\sqrt{\rho_\beta}\, G_{\mathrm{l}}^*), \\ a_{\beta,\mathrm{r}}(G) &= a_{\mathrm{r}}(\sqrt{1+\rho_\beta}\, G_{\mathrm{r}}) + a_{\mathrm{l}}^*(\sqrt{\rho_\beta}\, G_{\mathrm{r}}^*) \end{aligned} \tag{2.29}$$

and creation operators as their adjoints. Note that as a function of G, the right annihilation operator is linear. This fits the interpretation that the right annihilation operator annihilates a hole, i.e. creates a photon. At zero temperature, $\beta = \infty$, this reduces to $a_{\infty,\mathrm{l}}^\#(G) = a_{\mathrm{l}}^\#(G_{\mathrm{l}})$ and $a_{\infty,\mathrm{r}}^\#(G) = a_{\mathrm{r}}^\#(G_{\mathrm{r}})$.

The self-adjoint operator formally corresponding to $\phi(G)$ at positive temperature under the representation $\pi_{\beta,\mathrm{l}}^{\mathrm{PF}}$, cf. (1.16), can now be explicitly written as

$$\phi_{\beta,\mathrm{l}}^{\mathrm{PF}}(G) = \frac{1}{\sqrt{2}} \big(a_{\beta,\mathrm{l}}^*(G) + a_{\beta,\mathrm{l}}(G) \big). \tag{2.30}$$

We proceed to identify a suitable G_{L} useful for reformulating $\phi_{\beta,\mathrm{l}}^{\mathrm{PF}}(G)$ in terms of left and right fields $\phi_{\mathrm{l}}(G_{\mathrm{L}})$ and $\phi_{\mathrm{r}}(G_{\mathrm{L}})$. Here

$$\phi_{\mathrm{l/r}}(G_{\mathrm{L}}) = \frac{1}{\sqrt{2}}\big(a_{\mathrm{l/r}}^*(G_{\mathrm{L}}) + a_{\mathrm{l/r}}(G_{\mathrm{L}})\big)$$

are defined in terms of left and right annihilation and creation operators (2.27).

All annihilation and creation operators, having densely defined adjoints, are closable on $\mathcal{C}^{\mathrm{L}}$. We observe that the left sets of annihilation and creation operators form representations of CCR, non-Fock at positive temperature. As for the right representations, they are also representations of CCR, but with the roles of annihilation and creation operators reversed. The left and right operators, at the same inverse temperature, commute. As for the field operators, they are all by the usual analytic vector argument essentially self-adjoint on $\mathcal{C}^{\mathrm{L}}$. Left and right fields, at the same inverse temperature, commute.

At finite inverse temperature β we introduce $G_{\beta,\mathrm{l}}, G_{\beta,\mathrm{r}} \in L^2(\mathbb{R}^3; \mathcal{B}(\mathcal{K} \otimes \mathcal{K}))$ by the prescription

$$G_{\beta,\mathrm{l}} = \sqrt{1 + \rho_\beta}\, G_{\mathrm{l}} - \sqrt{\rho_\beta}\, G_{\mathrm{r}}^* \quad \text{and} \quad G_{\beta,\mathrm{r}} = \sqrt{1 + \rho_\beta}\, G_{\mathrm{r}} - \sqrt{\rho_\beta}\, G_{\mathrm{l}}^*. \tag{2.31}$$

Note that G_{l} and G_{r} are the zero temperature limits of $G_{\beta,\mathrm{l}}$ and $G_{\beta,\mathrm{r}}$, and $J_{\mathrm{p}}G_{\beta,\mathrm{l}}(k) = G_{\beta,\mathrm{r}}(k)J_{\mathrm{p}}$. The interaction at positive temperature is $W_\beta(G)$ where

$$W_\beta(G) := \phi_{\beta,\mathrm{l}}^{\mathrm{PF}}(G) - \phi_{\beta,\mathrm{r}}^{\mathrm{PF}}(G) = \phi_{\mathrm{l}}(G_{\beta,\mathrm{l}}) - \phi_{\mathrm{r}}(G_{\beta,\mathrm{r}}). \tag{2.32}$$

The two expressions can easily be seen to coincide on $\mathcal{C}^{\mathrm{L}}$, a common domain of essential self-adjointness.

The positive temperature Liouvillean is thus densely defined, a priori on $\mathcal{C}^{\mathrm{L}}$, as the operator sum

$$L_\beta = L_0 + W_\beta(G) = L_\infty + I_\beta(G),$$

where

$$I_\beta(G) = \phi_{\mathrm{l}}(G_{\beta,\mathrm{l}} - G_{\mathrm{l}}) - \phi_{\mathrm{r}}(G_{\beta,\mathrm{r}} - G_{\mathrm{r}}). \tag{2.33}$$

That the Liouvilleans L_β, for $0 < \beta < \infty$, are essentially self-adjoint on $\mathcal{C}^{\mathrm{L}}$ was proved in [39, Lemma 3.2], cf. also [10, 15, 47], using Nelson's commutator theorem [59, Thm. X.37]. This requires that G can absorb a power of the dispersion $|k|$, which is the source of the $\delta_{\alpha,0}$ term in the ultraviolet part of condition **(LGn)**. We warn the reader that the domain of L_β may depend on both β and G, an issue that complicates the analysis of the operator. Proposition 2.7 2 below remedies this issue somewhat.

We write $N^{\mathrm{L}} = N \otimes \mathbb{1}_{\mathcal{F}} + \mathbb{1}_{\mathcal{F}} \otimes N$ for the number operator on $\mathcal{F} \otimes \mathcal{F}$, and as for N we use the same notation to denote $\mathbb{1}_{\mathcal{K}} \otimes N \otimes \mathbb{1}_{\mathcal{K}} \otimes N$.

We recall the modular conjugation J from (1.17), which we here express in terms of the conjugation $\mathbf{C}$ from the end of the last subsection:

$$J = (\mathbf{C} \otimes \mathbf{C})\,\mathcal{E}, \tag{2.34}$$

where $\mathcal{E}$ is the exchange operator defined on simple tensors by $\mathcal{E}(\psi \otimes \varphi) = \varphi \otimes \psi$. Here $\psi, \varphi \in \mathcal{H}$. Clearly $JL_\infty J = -L_\infty$. Indeed, the identity holds on $\mathcal{C}^{\mathrm{L}}$ and extends to $\mathcal{D}(L_\infty)$ since $\mathcal{C}^{\mathrm{L}}$ is an operator core for L_∞.

Computing as an identity first on $\mathcal{C}^{\mathrm{L}}$ we find

$$J\phi_{\mathrm{l}}(G_{\beta,\mathrm{l}})J = \phi_{\mathrm{r}}(G_{\beta,\mathrm{r}}) \quad \text{and} \quad J\phi_{\beta,\mathrm{l}}^{\mathrm{PF}}(G)J = \phi_{\beta,\mathrm{r}}^{\mathrm{PF}}(G),$$

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and hence
$$JL_\beta J = -L_\beta.$$

As above one should first verify the identities on $\mathcal{C}^L$ and extend by continuity to $\mathcal{D}(L_\beta)$. Consequently, we observe that the spectrum and pure point spectrum of L_β is reflection symmetric around 0. Furthermore the spectral resolution E^β associated with L_β satisfies $E^\beta(B) = JE^\beta(-B)J$ and hence the absolutely and singular continuous spectra of L_β are also reflection symmetric.

Lemma 2.6. *Let $G \in L^2(\mathbb{R}^3; \mathrm{M}_\nu(\mathbb{C}))$ and $\psi \in \mathcal{C}^L$.*

$$\|W_\beta(G)\psi\| \le \big(\|G\| + \frac{2}{\sqrt{\beta}}\||k|^{-\frac{1}{2}}G\|\big)\big(2^{\frac{3}{2}}\|\psi\| + 2^{\frac{1}{2}}\|\sqrt{N^L}\psi\|\big), \qquad (2.35)$$

$$|\langle\psi, W_\beta(G)\psi\rangle| \le \sigma\langle\psi, N^L\psi\rangle + \sigma^{-1}\big(\|G\| + \frac{2}{\sqrt{\beta}}\||k|^{-\frac{1}{2}}G\|\big)^2\|\psi\|^2. \qquad (2.36)$$

Proof. Use the representation of $W_\beta(G)$ as the difference of two zero temperature fields, cf. (2.32), together with Lemma 2.1 1 and the bounds

$$\begin{aligned}\|G_{\beta,\mathrm{l/r}}\| &\le \|\sqrt{1 + \rho_\beta}G\| + \|\sqrt{\rho_\beta}G\| \\ &\le \|G\| + 2\|\sqrt{\rho_\beta}G\| \\ &\le \|G\| + \frac{2}{\sqrt{\beta}}\||k|^{-\frac{1}{2}}G\|.\end{aligned}$$

The last inequality follows from

$$|k|\rho_\beta = \frac{|k|}{\mathrm{e}^{\beta|k|} - 1} \le \beta^{-1}.$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

We end the subsection with a proposition that permits us to work effectively with standard Liouvilleans, despite domain problems. Its proof follows closely arguments from [19], establishing similar statements for technically related operators. The proposition involves the $C^1_{\mathrm{Mo}}(A)$ class introduced in Appendix A.3.

Proposition 2.7. *Suppose* **(LG0)**. *The following holds*

1. *$N^L \in C^1_{\mathrm{Mo}}(L_\beta)$ and the operator $[N^L, L_\beta]^\circ$ extends from $\mathcal{D}(N^L)$ by continuity to an element of $\mathcal{B}(\mathcal{D}(\sqrt{N^L}); \mathcal{H}^L)$.*
2. *$\mathcal{D}(N^L) \cap \mathcal{D}(L_\beta)$ does not depend on β, nor on G.*
3. *$\mathcal{C}^L$ is dense in $\mathcal{D}(N^L) \cap \mathcal{D}(L_\beta)$ with respect to the intersection topology.*

Proof. To establish 1 we argue as in the verification of [19, Cond. 2.1 (2), cf. Sect. 5.5].

First observe that N^L and L_0 commute such that we can compute as a form on the core $\mathcal{C}^L$

$$\big[(N^L + 1)^{-1}, L_\beta\big] = (N^L + 1)^{-1}W_\beta(G) - W_\beta(G)(N^L + 1)^{-1}.$$

The right-hand side extends to a bounded operator, and since $\mathcal{C}^L$ is dense in $\mathcal{D}(L_\beta)$, the form $[(N^L + 1)^{-1}, L_\beta]$ defined on $\mathcal{D}(L_\beta)$ extends by continuity to a bounded form on $\mathcal{H}^L$, coinciding with the closure of the right-hand side as a form on $\mathcal{C}^L$. Hence $N^L \in C^1(L_\beta)$, cf. Definition A.1.

Having established that N^{L} is of class $C^1(L_\beta)$ we know that $[N^{\mathrm{L}}, L_\beta]$ extends from the intersection domain $\mathcal{D}(N^{\mathrm{L}}) \cap \mathcal{D}(L_\beta)$ to a bounded form on $\mathcal{D}(N^{\mathrm{L}})$. Hence, to compute this form it suffices to compute it on a core of N^{L}. Compute as a form on $\mathcal{C}^{\mathrm{L}}$

$$\imath[N^{\mathrm{L}}, L_\beta] = W_\beta(\imath G), \tag{2.37}$$

which due to (2.35) extends from $\mathcal{C}^{\mathrm{L}}$ to $\mathcal{B}(\mathcal{D}(\sqrt{N^{\mathrm{L}}}); \mathcal{H}^{\mathrm{L}})$. This proves 1.

As for 2 we follow the proof of [19, Lemma 5.15]. Let $T_0 = L_0 + \imath(N^{\mathrm{L}} + 1)$. Since L_0 and N^{L} commute we clearly have $\mathcal{D}(T_0) = \mathcal{D}(L_0) \cap \mathcal{D}(N^{\mathrm{L}}) =: \mathcal{D}_0$.

We now construct $L_\beta + \imath(N^{\mathrm{L}} + 1)$ in two different ways. First define $\widehat{L} = L_0 + W_\beta(G)$ as a symmetric operator on $\mathcal{D}_0$. Then $T_0 + W_\beta(G) = \widehat{L} + \imath(N^{\mathrm{L}} + 1) =: T_1$ is by [59, Corollary to Thm. X.48] a closed operator on $\mathcal{D}_0$. Here we used Lemma 2.35 again. Conversely we can use Proposition A.10 to construct $T_2^\pm = L_\beta \pm \imath(N^{\mathrm{L}} + 1)$ as closed operators on $\mathcal{D}_\beta = \mathcal{D}(L_\beta) \cap \mathcal{D}(N^{\mathrm{L}})$ with $T_2^{+*} = T_2^-$. Since $\mathcal{C}^{\mathrm{L}} \subset \mathcal{D}_0$ we find that $\widehat{L}$ is a symmetric extension of $L_{\beta_{|\mathcal{C}^{\mathrm{L}}}}$. Hence, $\mathcal{C}^{\mathrm{L}}$ being a core for L_β, we find that $\widehat{L} \subset L_\beta$. This implies that $T_1 \subset T_2^+ =: T_2$. Since $T_2^\pm$ are both accretive we find that T_2 generates a contraction semigroup. To conclude the proof we only need to show that the $\rho(T_1) \cap \rho(T_2) \neq \emptyset$. But this follows from the Hille-Yosida theorem, cf. [59, Thm. X.47a].

Finally we turn to 3. From what was just proved, together with the closed graph theorem, we conclude that it suffices to prove that $\mathcal{C}^{\mathrm{L}}$ is dense in $\mathcal{D}_0$ with respect to the norm

$$\|\psi\|_0 = \|N^{\mathrm{L}}\psi\|_{\mathcal{H}^{\mathrm{L}}} + \|L_0\psi\|_{\mathcal{H}^{\mathrm{L}}} + \|\psi\|_{\mathcal{H}^{\mathrm{L}}}.$$

Since L_0 and N^{L} commute it suffices to show that one can approximate $\psi \in \mathcal{D}_0$ with $\psi = \mathbb{1}[N^{\mathrm{L}} \leq n]\psi$, for some n. Similarly, since L_0 and N^{L} commute with $\Gamma_R := \mathbb{1}_{\mathcal{K}} \otimes \Gamma(\mathbb{1}[|k| \leq R]) \otimes \mathbb{1}_{\mathcal{K}} \otimes \Gamma(\mathbb{1}[|k| \leq R])$, it suffices to approximate states ψ, non-zero in finitely many particle sectors, and satisfying $\Gamma_R\psi = \psi$, for some $R > 0$.

Let $\{\varphi_n\}_{n \in \mathbb{N}} \subset \mathcal{C}^{\mathrm{L}}$ be a sequence with $\|\psi - \varphi_n\|_{\mathcal{H}^{\mathrm{L}}} \to 0$ for $n \to \infty$. Let $\chi \in C_0^\infty(\mathbb{R}^3)$ satisfy $0 \leq \chi \leq 1$, $\chi(k) = 1$ for $|k| \leq R$, and $\chi(k) = 0$ for $|k| \geq R+1$. Then $\Gamma_\chi := \mathbb{1}_{\mathcal{K}} \otimes \Gamma(\chi) \otimes \mathbb{1}_{\mathcal{K}} \otimes \Gamma(\chi)$ preserves $\mathcal{C}^{\mathrm{L}}$ and $\|\psi - \Gamma_\chi\varphi_n\|_{\mathcal{H}^{\mathrm{L}}} \to 0$ for $n \to \infty$.

Now that both ψ and $\Gamma_\chi\varphi_n$ only have finitely many non-zero components all supported inside a box of side length $R + 1$, one can easily verify that

$$\|\psi - \Gamma_\chi\varphi_n\|_0 \to 0, \quad \text{for} \quad n \to \infty.$$

This completes the proof. $\qquad\square$

We remark that L_β is presumably *not* of class $C^1(N^{\mathrm{L}})$, cf. Lemma 2.2.

2.3 Jakšić-Pillet Gluing

We proceed to discuss a unitarily equivalent form of the Liouvillean obtained by the so-called Jakšić-Pillet gluing procedure, cf. [15, 39].

But first we pass to polar coordinates on the Hamiltonian level. Define a unitary transform $T_1 \colon \mathfrak{h} \to \tilde{\mathfrak{h}}_1 := L^2([0, \infty)) \otimes L^2(S^2)$ by the prescription

$$(T_1 f)(\omega, \Theta) = \omega f(\omega\Theta).$$

Denote by $\widetilde{\mathcal{F}}_1 = \Gamma(\tilde{\mathfrak{h}}_1)$ the Fock space in polar coordinates. The subscript 1 is for later use and refers to the left component in the tensor product $\mathcal{H}^{\mathrm{L}} = \mathcal{H} \otimes \mathcal{H}$. The

twiddle indicates an object represented in polar coordinates for the Hamiltonian, and after gluing for the Liouvillean.

The coupling in polar coordinates becomes

$$\widetilde{G}(\omega, \Theta) := \omega G(\omega\Theta)$$

and the Hamiltonian takes the form

$$\widetilde{H} = \big(\mathbb{1}_{\mathcal{K}} \otimes \Gamma(T_{\mathrm{l}})\big) H \big(\mathbb{1}_{\mathcal{K}} \otimes \Gamma(T_{\mathrm{l}})^*\big) = K \otimes \mathbb{1}_{\widetilde{\mathcal{F}}_{\mathrm{l}}} + \mathbb{1}_{\mathcal{K}} \otimes \mathrm{d}\Gamma(\omega) + \phi(\widetilde{G}),$$

a priori as an identity on $\mathcal{K} \otimes \Gamma_{\mathrm{fin}}(C_0^\infty([0, \infty) \otimes C^\infty(S^2))$ and extended to $\mathcal{D}(\widetilde{H}) = \mathcal{D}(\mathbb{1}_{\mathcal{K}} \otimes \mathrm{d}\Gamma(\omega))$ by continuity.

To deal with the standard Liouvillean we similarly need a map $T_{\mathrm{r}} \colon \mathfrak{h} \to \tilde{\mathfrak{h}}_{\mathrm{r}} := L^2((-\infty, 0]) \otimes L^2(S^2)$, defined by $(T_{\mathrm{r}}f)(\omega, \Theta) = (T_{\mathrm{l}}f)(-\omega, \Theta)$. Put $\widetilde{\mathcal{F}}_{\mathrm{r}} = \Gamma(\tilde{\mathfrak{h}}_{\mathrm{r}})$. This sets up a unitary transformation

$$T \colon \mathfrak{h} \oplus \mathfrak{h} \to \tilde{\mathfrak{h}} := L^2(\mathbb{R}) \otimes L^2(S^2),$$

by the construction

$$(T(f, g))(\omega, \Theta) = \mathbb{1}[\omega \geq 0](T_{\mathrm{l}}f)(\omega, \Theta) + \mathbb{1}[\omega \leq 0](T_{\mathrm{r}}g)(\omega, \Theta).$$

Using the canonical identification $I \colon \Gamma(\mathfrak{h} \oplus \mathfrak{h}) \to \mathcal{F} \otimes \mathcal{F}$ we construct a unitary map

$$\mathcal{U} \colon \mathcal{H}^{\mathrm{L}} \to \widetilde{\mathcal{H}}^{\mathrm{L}} := \mathcal{K} \otimes \mathcal{K} \otimes \widetilde{\mathcal{F}},$$

where $\widetilde{\mathcal{F}} = \Gamma(L^2(\mathbb{R}) \otimes L^2(S^2))$. The map $\mathcal{U}$ is defined on simple tensors by

$$\mathcal{U}(u \otimes \eta \otimes v \otimes \xi) = u \otimes v \otimes \Gamma(T)I^*(\eta \otimes \xi)$$

and extended to $\mathcal{H}^{\mathrm{L}}$ by linearity and continuity. Here $u, v \in \mathcal{K}$ and $\xi, \eta \in \mathcal{F}$. As an alternative core we take

$$\widetilde{\mathcal{C}}^{\mathrm{L}} = \mathcal{K} \otimes \mathcal{K} \otimes \Gamma_{\mathrm{fin}}\big(C_0^\infty(\mathbb{R}) \otimes C^\infty(S^2)\big). \tag{2.38}$$

In the new coordinate system, we can write the interaction $W_\beta(G)$ as a field operator as follows. First, the zero temperature interaction is

$$\widetilde{G}_\infty := \mathbb{1}[|\omega \geq 0]\widetilde{G}_{\mathrm{l}}(\omega, \Theta) - \mathbb{1}[\omega \leq 0]\widetilde{G}_{\mathrm{r}}(-\omega, \Theta), \tag{2.39}$$

where $\widetilde{G}_{\mathrm{l/r}}(\omega, \Theta) = \omega G_{\mathrm{l/r}}(\omega\Theta)$, cf. (2.28). With this construction we have the identity $\mathcal{U}(\phi_{\mathrm{l}}(G_{\mathrm{l}}) - \phi_{\mathrm{r}}(G_{\mathrm{r}}))\mathcal{U}^* = \phi(\widetilde{G}_\infty)$. The computation is easily done on $\widetilde{\mathcal{C}}^{\mathrm{L}}$ and extended by continuity to $\mathcal{D}(\phi(\widetilde{G}))$. At finite temperature, the interaction reads

$$\widetilde{G}_\beta(\omega, \Theta) := \sqrt{1 + \widetilde{\rho}_\beta}\, \widetilde{G}_\infty + \sqrt{\widetilde{\rho}_\beta}\, \widetilde{G}_{\infty, \mathcal{R}}^*, \tag{2.40}$$

where $\widetilde{G}_{\infty, \mathcal{R}}(\omega, \Theta) = \widetilde{G}_\infty(-\omega, \Theta)$ is the reflected glued coupling, and

$$\widetilde{\rho}_\beta(\omega, \Theta) = \rho_\beta(\omega\Theta) = \frac{1}{\mathrm{e}^{\beta|\omega|} - 1}.$$

Recalling (2.31), we observe that we similarly have

$$\mathcal{U}W_\beta(G)\mathcal{U}^* = \mathcal{U}(\phi_{\mathrm{l}}(G_{\beta,\mathrm{l}}) - \phi_{\mathrm{r}}(G_{\beta,\mathrm{r}}))\mathcal{U}^* = \phi(\widetilde{G}_\beta).$$

As observed in [39] we have the following alternative representation of $\widetilde{G}_\beta$

$$\widetilde{G}_\beta = \left(\frac{\omega}{1 - \mathrm{e}^{-\beta\omega}} \right)^{\frac{1}{2}} \widehat{G}_\mathrm{l} - \left(\frac{\omega}{\mathrm{e}^{\beta\omega} - 1} \right)^{\frac{1}{2}} \widehat{G}_\mathrm{r}, \tag{2.41}$$

where

$$\begin{aligned}
\widehat{G}_\mathrm{l}(\omega, \Theta) &= \left(\mathbb{1}[\omega \geq 0]\sqrt{\omega}\, G(\omega\Theta) + \mathbb{1}[\omega \leq 0]\sqrt{-\omega}\, G(-\omega\Theta)^* \right) \otimes \mathbb{1}_\mathcal{K} \\
\widehat{G}_\mathrm{r}(\omega, \Theta) &= \mathbb{1}_\mathcal{K} \otimes \left(\mathbb{1}[\omega \geq 0]\sqrt{\omega}\, \overline{G(\omega\Theta)} + \mathbb{1}[\omega \leq 0]\sqrt{-\omega}\, \overline{G(-\omega\Theta)}^* \right).
\end{aligned} \tag{2.42}$$

This form of the interaction mirrors the Araki-Woods representation (2.29).

Remark 2.8. The representation (2.40) allows us to easily observe that under the assumption **(LGn)**, the ultraviolet part of $\widetilde{G}_\beta$ and its first n derivatives are square integrable, whereas (2.41) allows us to conclude the same for the infrared region. This sets up an application of Lemma 2.1 1, which holds true for any one-particle space $\mathfrak{h}$, not just $L^2(\mathbb{R}^3)$. $\diamond$

We can now write down the standard Liouvillean in the new coordinate system as

$$\widetilde{L}_\beta = L_\mathrm{p} \otimes \mathbb{1}_{\widetilde{\mathcal{F}}} + \mathbb{1}_{\mathcal{K}\otimes\mathcal{K}} \otimes \widetilde{H}_\mathrm{ph} + \phi(\widetilde{G}_\beta),$$

with $\widetilde{H}_\mathrm{ph} = \mathrm{d}\Gamma(\omega)$. Note that ω denotes both a real number and multiplication by the identity function in $L^2(\mathbb{R})$. Here L_p is the standard Liouvillean for the small quantum system, cf. (1.4).

Again, by Nelson's commutator theorem, $\widetilde{L}_\beta$ is essentially self-adjoint on $\widetilde{\mathcal{C}}^\mathrm{L}$. We observe that L_β and $\widetilde{L}_\beta$ are unitarily equivalent through $\mathcal{U}$. As an identity on $\widetilde{\mathcal{C}}^\mathrm{L}$ we have $\widetilde{L}_\beta = \widetilde{L}_\infty + \phi(\widetilde{G}_\beta - \widetilde{G}_\infty)$, with $\widetilde{L}_\infty = \widetilde{L}_0 + \phi(\widetilde{G}_\infty)$ and $\widetilde{L}_0 = L_\mathrm{p} \otimes \mathbb{1}_{\widetilde{\mathcal{F}}} + \mathbb{1}_{\mathcal{K}\otimes\mathcal{K}} \otimes \widetilde{H}_\mathrm{ph}$. These operators are also essentially self-adjoint on $\widetilde{\mathcal{C}}^\mathrm{L}$ and their closures are unitarily equivalent with the appropriate untwiddled objects.

In the glued coordinate system we write $\widetilde{N} = \mathcal{U}N^\mathrm{L}\mathcal{U}^* = \mathrm{d}\Gamma(\mathbb{1}_{\widetilde{\mathfrak{h}}})$, where the second quantization is here performed in $\widetilde{\mathcal{F}}$.

The statements 1 and 2 in the following corollary to Proposition 2.7 are an immediate consequence of Proposition 2.7 1 and 2. The item 3 however is not, but it can be proved by an argument identical to the one employed at the end of the proof above.

Corollary 2.9. *Suppose* **(LG0)**. *The following holds*

1. $\widetilde{N} \in C^1_\mathrm{Mo}(\widetilde{L}_\beta)$ *and the operator* $[\widetilde{N}, \widetilde{L}_\beta]^\circ$ *extends from* $\mathcal{D}(\widetilde{N})$ *by continuity to an element of* $\mathcal{B}(\mathcal{D}(\sqrt{\widetilde{N}}); \widetilde{\mathcal{H}}^\mathrm{L})$.

2. $\mathcal{D}(\widetilde{N}) \cap \mathcal{D}(\widetilde{L}_\beta)$ *does not depend on* β, *nor on* G.

3. $\widetilde{\mathcal{C}}^\mathrm{L}$ *is dense in* $\mathcal{D}(\widetilde{N}) \cap \mathcal{D}(\widetilde{L}_\beta)$ *with respect to the intersection topology.*

We remark that it is a consequence of Proposition 2.7 and the above corollary that, supposing **(LG0)**, the resolvents of L_β and $\widetilde{L}_\beta$ are strongly continuous in $\beta \in (0, \infty]$ and G, using the norm $\|G\|_0' = \|(1 + |k|^{-1/2})G\|$. Indeed, it suffices to prove strong convergence on $\mathcal{D}(\widetilde{N})$ where we compute

$$(\widetilde{L}_\beta(G) - z)^{-1} - (\widetilde{L}_{\beta'}(G') - z)^{-1} = (\widetilde{L}_{\beta'}(G') - z)^{-1}\phi(\widetilde{G}_\beta - \widetilde{G}'_{\beta'})(\widetilde{L}_\beta(G) - z)^{-1}.$$

Here we used Proposition A.13 2, cf. Corollary 2.9 1, which ensures that $(\widetilde{L}_\beta(G) - z)^{-1} \colon \mathcal{D}(\widetilde{N}) \to \mathcal{D}(\widetilde{N})$. The result now follows by observing that

$$\lim_{(\beta', G') \to (\beta, G)} \left\| \widetilde{G}_\beta - \widetilde{G}'_{\beta'} \right\| = 0.$$

Norm continuity, even in β, of the resolvent is probably false but an argument is lacking. This means that while $\sigma(L_\infty) = \mathbb{R}$, cf. [58, Thm. VIII.33], we cannot a priori exclude that the spectrum of L_β could collapse for $\beta < \infty$ to become a proper subset of $\mathbb{R}$, cf. the discussion around [58, Thm. VIII.24].

Instead we proceed as for the Hamiltonian, via a pull through formula:

Proposition 2.10. *Suppose* **(LG0)**. *For any* $\beta > 0$, $z \in \mathbb{C} \backslash \mathbb{R}$ *and* $\psi \in \mathcal{D}(\sqrt{\widetilde{N}})$ *we have as an* $L^2(\mathbb{R} \times S^2; \widetilde{\mathcal{H}}^{\mathrm{L}})$*-identity*

$$a(\omega, \Theta)(\widetilde{L}_\beta - z)^{-1}\psi = (\widetilde{L}_\beta + \omega - z)^{-1}a(\omega, \Theta)\psi - (\widetilde{L}_\beta + \omega - z)^{-1}(\widetilde{G}_\beta(\omega, \Theta) \otimes \mathbb{1}_{\widetilde{\mathcal{F}}})\psi.$$

We omit the proof of Proposition 2.10 since it is verbatim the same as for Proposition 2.3, keeping in mind Corollary 2.9. In particular the consequence that resolvents of $\widetilde{L}_\beta$ preserves $\mathcal{D}(\widetilde{N})$, hence also $\mathcal{D}(\sqrt{\widetilde{N}})$, and that $\widetilde{\mathcal{C}}^{\mathrm{L}}$ is an operator core for $\widetilde{L}_\beta$. We are now almost ready to prove the following theorem, which seems to be new and establishes a widely expected result as a fact.

Theorem 2.11. *Suppose* **(LG0)**. *For any* $\beta > 0$ *we have* $\sigma(L_\beta) = \sigma(\widetilde{L}_\beta) = \mathbb{R}$.

We postpone the proof of the above theorem to Subsect. 3.4, where a missing ingredient will be introduced. The two HVZ-type theorems, Theorems 2.5 and 2.11, will play no role in the notes apart from clarifying the general spectral picture.

Remark 2.12. Our results on the standard Liouvillean will mostly be proved in the Jakšić-Pillet glued coordinates. Since only ω-derivatives will play a role, this allows us to formulate slightly weaker assumptions using $\widetilde{G}_\beta$ instead of G. This improvement is in general largely irrelevant, hence the present formulation with **(LGn)**.

More importantly, for couplings G on a special form, one can due to the representation (2.41) allow for interactions at positive temperature far more singular than what is permitted by **(LGn)**. To make this precise assume G takes the form

$$G(k) = |k|^{-1/2} g(k) G_0, \tag{2.43}$$

where $G_0 \in \mathrm{M}_\nu(\mathbb{C})$ is self-adjoint $G_0^* = G_0$, and $g \colon \mathbb{R}^3 \to \mathbb{R}$. Define

$$\hat{g}(\omega, \Theta) = \mathbb{1}[\omega \geq 0]g(\omega\Theta) + \mathbb{1}[\omega \leq 0]g(-\omega\Theta).$$

Then we can represent $\widehat{G}_{\mathrm{l}} = \hat{g}G_0$ and $\widehat{G}_{\mathrm{r}} = \hat{g}\overline{G}_0$, cf. (2.42). Hence we see that differentiability of $\widetilde{G}_\beta$ is governed by that of $\hat{g}$. For the spin-boson model g is a form factor (or ultraviolet cutoff), e.g. constant near 0 or perhaps of the form $\mathrm{e}^{-k^2/\Lambda^2}$ to take some popular choices. Here $\hat{g}$ will be constant across the singularity at $\omega = 0$, or equal to $\mathrm{e}^{-\omega^2/\Lambda^2}$ for the other choice.

For models of the form (2.43) we can reformulate replacements for **(LGn)**. Let $n \in \mathbb{N}_0$. There exists $\hat{g} \colon \mathbb{R} \times S^2 \to \mathbb{C}$ and $G_0 \in \mathrm{M}_\nu(\mathbb{C})$ self-adjoint such that $\hat{g}$ admits n distributional ω derivatives in $L^1_{\mathrm{loc}}(\mathbb{R} \times S^2)$ and such that

$$
\begin{aligned}
& G(\omega\Theta) = |\omega|^{-\frac{1}{2}} \hat{g}(\omega, \Theta) G_0 \\
\textbf{(LGn}') \quad & \forall \omega\Theta \in \mathbb{R}^3, |\omega| \leq 1, \quad \text{and} \quad j \leq n: \quad |\partial_\omega^j \hat{g}(\omega, \Theta)| \leq C|\omega|^{n-1+\mu-j} \\
& \forall \omega\Theta \in \mathbb{R}^3, |\omega| \geq 1, \quad \text{and} \quad j \leq n: \quad |\partial_\omega^j \hat{g}(\omega, \Theta)| \leq C|\omega|^{-1-\delta_{j,0}-\mu}.
\end{aligned}
$$

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This type of condition was used in [16, 23].

Finally we remark that it was observed and utilized in [23], that the Jakšić-Pillet gluing is not canonical in that one can glue the two reservoirs together at $\omega = 0$, twisting one of them with a phase. This allows one to consider $\hat{g}$ as complex valued and then pick the gluing phase such that $\hat{g}(\omega\Theta)$ and $\overline{\hat{g}(-\omega\Theta)}$ fit together seamlessly across $\omega = 0$. In fact, one can in this way also allow for singular behavior of the form $|k|^{1/2}$ at zero, and not just $|k|^{-1/2}$. This would require an extra twist by the angle π corresponding to a sign change across zero. $\diamond$

2.4 Multiple Reservoirs

We have made a choice, in the name of concreteness, to focus on finite dimensional quantum systems coupled to a massless scalar field (in three dimensions) and their thermal Liouvilleans.

Our methods and indeed theorems however have validity beyond this particular choice. We single out here the case of multiple reservoirs at possibly different inverse temperatures $\vec{\beta} = (\beta_1, \ldots, \beta_q)$.

The easiest way to observe that the results of these notes carry over to the case of multiple reservoirs is to replace $\mathfrak{h} = L^2(\mathbb{R}^3)$ by $L^2(\mathbb{R}^3 \times \{1, \ldots, q\}) \sim \mathfrak{h}^q$, q being the number of reservoirs. The dispersion becomes $\omega(k, j) = |k|$ (or $|k|\mathbb{1}_{\mathbb{C}^q}$). Given q couplings $G_1, \ldots, G_q$ all satisfying the same sets of conditions, one can construct a coupling for the multi-reservoir system by setting $G(k, j) = G_j(k)$.

As for the standard Liouvillean one should replace $G_{\beta,\mathrm{l/r}}$ from (2.31) by the functions $G_{\vec{\beta},\mathrm{l/r}}(\cdot, j) = G_{\beta_j,\mathrm{l/r}}(\cdot, j)$. Similarly for $\widetilde{G}_{\vec{\beta}}$.

Weak-coupling as well as high and low-temperature results remain valid if all coupling, respectively temperatures, are taken into the same regime.

Only one type of result here does not extend to the case of multiple reservoirs, and that is the existence/non-existence results for eigenvalues of L_β discussed in Subsect. 3.4, which make critical use of the modular structure of the thermal Liouvilleans. In fact, if two inverse temperatures are distinct, at weak coupling and under a suitable non-triviality condition on G one has $\sigma_{\mathrm{pp}}(L_{\vec{\beta}}) = \emptyset$, cf. [16, Thm. 7.17]. this reflects the fact that an atom coupled to multiple reservoirs at different temperatures does not have an invariant state.

One could also replace the thermal density ρ_β by other densities and a number of our results remain valid. However, the reader doing that would have to reformulate the condition (**LGn**) where the $1/|k|$ singularity of ρ_β is built in. As for our low-temperature result, these would have to be translated into "small" density statements. The reader can consult [16, 17] for discussions of other models.

Finally we remark that essentially what we exploit is the Jakšić-Pillet glued representation of the standard Liouvillean and presumably one could rephrase everything in this abstract setup. See also [16, Sect. 8].

The reader can consult [42, 41, 49, 50] for papers devoted to multiple reservoirs. Three of these papers consider also the non-selfadjoint C-Liouvillean, which seems to be a more natural object when considering non-stationary steady states.

2.5 Open Problems II

There are not that many serious problems pertaining to the material from this section. We did mention two related conjectures regarding the standard Liouvillean, while not

in itself of great interest, resolving them would serve to clarify the picture:

Problem 2.1. Clarify to what extend the domain of the standard Liouvillean L_β is β and G dependent.

Problem 2.2. Verify that, as conjectured, the resolvents of the Liouvillean are *not* norm continuous in β and G.

As a final topic, we discuss the ultraviolet singularity of the models. For e.g. the spin-boson model, the coupling G goes as $1/\sqrt{|k|}$ for large momenta, which is more singular than what we can deal with. It is well-known that the Nelson (and the polaron) model is renormalizable, but this is due to a regularizing effect stemming from the small system, in that the Laplacian allows for control of the ultraviolet contributions [3, 45, 53]. Indeed, we do not expect that the spin-boson model has a meaningful ultraviolet limit and it should not be a relevant question since it is a model describing low energy/momentum phenomena only. Having said that, it would still be undesirable if the choice of (a reasonable) cutoff would influence whether or not the Liouvillean has a unique invariant state or admits non-zero eigenvalues, and if it does have non-zero eigenvalues, will the point spectrum, being related to energy differences, have an ultraviolet limit. This is an underlying, and largely unexplored, issue that will not play a role in these notes.

3 Bound States

In this section we study the basic properties of bound states. The key is the following formal computation

$$\langle \psi, \imath[H, A]\psi \rangle = 0,$$

whenever ψ is a bound state for H and A is some auxiliary operator. Choosing A such that the commutator $\imath[H, A]$ contains a positive operator N and a remainder controllable either by H or some fractional power of N, will imply - at least formally - that ψ is in the form domain of N. In our case, the operator N will be the number operator N (or N^{L} for the Liouvillean).

It turns out to be a surprisingly delicate question to establish such a bound rigorously for the standard Liouvillean, but for the Hamiltonian it is fairly straightforward. The first argument of this type is for the Hamiltonian and is due to Skibsted [62], and for the Liouvillean it goes back to Fröhlich and Merkli [24], cf. also [25]. The result we present here for the Liouvillean improves on the theorem of Fröhlich and Merkli.

As a consequence of such number bounds we will be able to establish virial theorems for the Hamiltonian and the Liouvillean.

3.1 Number Bounds at Zero Temperature

As for A we make the choice

$$A = \mathrm{d}\Gamma(a),$$

with

$$a = \frac{1}{2}\left\{ \frac{k}{|k|} \cdot \nabla_k + \nabla_k \cdot \frac{k}{|k|} \right\}, \tag{3.44}$$

the generator of radial translations. Note that a should be viewed as the closure of a restricted to $C_0^\infty(\mathbb{R}^3 \backslash \{0\})$ and that a is a maximally symmetric operator, but not self-adjoint. Since H is not of class $C^1(A)$, we cannot directly make sense out of the formal computation above.

Instead we introduce a family of regularized conjugate operators $A_n = \mathrm{d}\Gamma(a_n)$ with

$$a_n = \frac{1}{2}\left\{ \frac{k}{\sqrt{|k|^2 + n^{-1}}} \cdot \nabla_k + \nabla_k \cdot \frac{k}{\sqrt{|k|^2 + n^{-1}}} \right\}.$$

The a_n's, constructed as closures from $C_0^\infty(\mathbb{R}^3)$, are self-adjoint and $H \in C^1(A_n)$ for all n, provide (**HG1**) is assumed. This construction goes back to Skibsted [62] and was used also in [28].

Let ψ be a bound state for H, i.e. $H\psi = E\psi$ for some $E \in \mathbb{R}$. It is now a consequence of the standard Virial Theorem, cf. Theorem A.6, that $\langle \psi, \imath[H, A_n]^\circ \psi \rangle = 0$. Computing the commutator we find

$$\imath[H, A_n]^\circ = \mathrm{d}\Gamma\Big(\frac{|k|}{\sqrt{|k|^2 + n^{-1}}} \Big) - \phi(\imath a_n G).$$

Note that assuming (**HG1**) we have $a_n G \in L^2(\mathbb{R}^3; M_\nu(\mathbb{C}))$. From the estimate (2.24), applied with $|k|/\sqrt{|k|^2 + n^{-1}}$ in place of $|k|$, we get

$$\imath[H, A_n]^\circ \geq \frac{1}{2}\mathrm{d}\Gamma\Big(\frac{|k|}{\sqrt{|k|^2 + n^{-1}}} \Big) - C \big\| (|k|^2 + n^{-1})^{\frac{1}{4}} |k|^{-\frac{1}{2}} a_n G \big\|^2.$$

To check for the finiteness and uniform boundedness of the norm on the right-hand side we write $a_n = \frac{k}{\sqrt{|k|^2 + n^{-1}}} \cdot \imath \nabla_k + \frac{1}{2} \operatorname{div}(k/\sqrt{k^2 + n^{-1}})$ and estimate

$$\big\|(|k|^2 + n^{-1})^{\frac{1}{4}} |k|^{-\frac{1}{2}} a_n G \big\| \leq \big\|\nabla G\big\| + C \big\| G/|k| \big\|, \tag{3.45}$$

for some n-independent constant C. Since $n \to |k|/\sqrt{|k|^2 + n^{-1}}$ is monotonously increasing towards 1 we conclude from Lebesgue's theorem on monotone convergence the following:

Theorem 3.1. *Suppose* **(HG1)**. *There exists a $C > 0$ such that for any normalized bound state $\psi \in \mathcal{H}$ of H we have $\psi \in \mathcal{D}(\sqrt{N})$ and*

$$\big\|\sqrt{N}\psi\big\| \leq C\big(\|\nabla G\| + \|G/|k|\|\big).$$

Since N commutes with the conjugation $\mathbf{C}$, we observe that the same theorem holds for bound states of $H^{\mathbf{c}}$.

That the constant C in the theorem above can be chosen uniformly in E, is a consequence of $\mathcal{K}$ being finite dimensional. For e.g. the confined Nelson model, this would be false since one will need a resolvent of K to bound the relevant aG. This is however a mute point, since we in Subsect. 4.3 will prove that H does not have high energy bound states!

In fact if one assumes in addition **(HG2)** one can do better and get $\psi \in \mathcal{D}(N)$ using [19]. This is however a much deeper result and will not play a role in these notes.

3.2 Number Bounds at Positive Temperature

For the standard Liouvillean L_∞ at zero temperature we observe that since eigenstates are of the form $\psi \otimes \varphi$, with ψ, φ eigenstates of H and $H^{\mathbf{c}}$ respectively, they are due to Theorem 3.1 automatically in the domain of $\mathcal{D}(\sqrt{N^{\mathrm{L}}})$. Hence, eigenstates of $\widetilde{L}_\infty$ are in the domain of $\sqrt{\widetilde{N}}$. The situation at positive temperature is a good deal more subtle.

We begin with a key technical lemma, which enables us to compute commutators. The proof follows closely a similar argument from [19, Proof of Cond. 2.1 (3)].

Before stating the lemma we need some notation. Let $m \in C^\infty(\mathbb{R})$ be real-valued and bounded with bounded derivatives. Put

$$\tilde{a}_m = \frac{1}{2} \left\{ m \frac{\mathrm{d}}{\mathrm{d}\omega} + \frac{\mathrm{d}}{\mathrm{d}\omega} m \right\} \otimes \mathbb{1}_{L^2(S^2)} \tag{3.46}$$

and

$$\widetilde{A}_m = \mathbb{1}_{\mathcal{K} \otimes \mathcal{K}} \otimes \mathrm{d}\Gamma(\tilde{a}_m). \tag{3.47}$$

We leave it to the reader to argue that $\tilde{a}_m$ and $\widetilde{A}_m$ are essentially self-adjoint on $C_0^\infty(\mathbb{R}) \otimes C^\infty(L^2(S^2))$ and $\widetilde{\mathcal{C}}^{\mathrm{L}}$, respectively. Recall from (2.38) the form of the core $\widetilde{\mathcal{C}}^{\mathrm{L}}$.

Lemma 3.2. *Suppose* **(LG1)**. *Then*

$$\big\langle \psi, \imath[\widetilde{L}_\beta, (\widetilde{A}_m - z)^{-1}] \varphi \big\rangle = -\big\langle \psi, (\widetilde{A}_m - z)^{-1} \widetilde{L}'_\beta (\widetilde{A}_m - z)^{-1} \varphi \big\rangle,$$

for all $\psi, \varphi \in \mathcal{D}(\sqrt{\widetilde{N}}) \cap \mathcal{D}(\widetilde{L}_\beta)$ and $z \in \mathbb{C}$ with $\operatorname{Im} z \neq 0$. Here

$$\widetilde{L}'_\beta = \mathbb{1}_{\mathcal{K} \otimes \mathcal{K}} \otimes \mathrm{d}\Gamma(m) - \phi(\imath \tilde{a}_m \widetilde{G}_\beta)$$

defined as a form on $\mathcal{D}(\sqrt{\widetilde{N}})$.

Remark 3.3. We first observe that the expression in the lemma makes sense. Since $\widetilde{A}_m$ and $\widetilde{N}$ commute, $(\widetilde{A}_m - z)^{-1}$ preserves the domain of $\sqrt{\widetilde{N}}$. By boundedness of m and m' together with Remark 2.8, we see that $\widetilde{L}'_\beta$ is well-defined as a form on $\mathcal{D}(\sqrt{\widetilde{N}})$.

Proof. By Corollaries 2.9 and A.12 it suffices to establish the desired form-identity on $\mathcal{D}(\widetilde{N}) \cap \mathcal{D}(\widetilde{L}_\beta)$.

Recall from Corollary 2.9 that $\mathcal{D}(\widetilde{N}) \cap \mathcal{D}(\widetilde{L}_\beta) = \mathcal{D}(\widetilde{N}) \cap \mathcal{D}(\widetilde{L}_0)$. On this domain $\widetilde{L}_\beta$ can be written as the operator sum $\widetilde{L}_0 + \phi(\widetilde{G}_\beta)$. Hence it suffices to prove that

$$\big\langle \psi, \imath[\widetilde{L}_0, (\widetilde{A}_m - z)^{-1}]\varphi \big\rangle = -\big\langle \psi, (\widetilde{A}_m - z)^{-1} \mathbb{1}_{\mathcal{K}\otimes\mathcal{K}} \otimes \mathrm{d}\Gamma(m)(\widetilde{A}_m - z)^{-1}\varphi \big\rangle,$$
$$\big\langle \psi, \imath[\phi(\widetilde{G}_\beta), (\widetilde{A}_m - z)^{-1}]\varphi \big\rangle = \big\langle \psi, (\widetilde{A}_m - z)^{-1}\phi(\imath\tilde{a}_m\widetilde{G}_\beta)(\widetilde{A}_m - z)^{-1}\varphi \big\rangle,$$
(3.48)

for all $\psi, \varphi \in \mathcal{D}(\widetilde{N}) \cap \mathcal{D}(\widetilde{L}_0)$ and $z \in \mathbb{C}$ with $\mathrm{Im}\,z \neq 0$. The second identity in (3.48) can easily be verified for $\psi, \varphi \in \widetilde{\mathcal{C}}^{\mathrm{L}}$ from which it extends by density since $\phi(\widetilde{G}_\beta)$ and $\phi(\imath\tilde{a}_m\widetilde{G}_\beta)$ are $\sqrt{\widetilde{N}}$-bounded, cf. Remark 2.8.

As for the first identity in (3.48) one should first observe that all objects preserve particle sectors, i.e. sectors with $\widetilde{N} = n$ for some n. Hence it suffices to establish the identity for ψ, φ being n-particle states. Observe that $\mathrm{d}\Gamma^{(n)}(\omega)$ is of class $C^1_{\mathrm{Mo}}(\mathrm{d}\Gamma^{(n)}(\tilde{a}_m))$, indeed; $\imath[\mathrm{d}\Gamma^{(n)}(\omega), \mathrm{d}\Gamma^{(n)}(\tilde{a}_m)]^\circ = \mathrm{d}\Gamma^{(n)}(m)$ is a bounded operator on $\mathcal{F}^{(n)}$, the n-particle sector. Hence the identity

$$\big\langle \psi, \mathbb{1}_{\mathcal{K}\otimes\mathcal{K}} \otimes \imath[\mathrm{d}\Gamma^{(n)}(\omega), (\mathrm{d}\Gamma^{(n)}(\tilde{a}_m) - z)^{-1}]\varphi \big\rangle$$
$$= -\big\langle \psi, \mathbb{1}_{\mathcal{K}\otimes\mathcal{K}} \otimes (\mathrm{d}\Gamma^{(n)}(\tilde{a}_m) - z)^{-1}\mathrm{d}\Gamma^{(n)}(m)(\mathrm{d}\Gamma^{(n)}(\tilde{a}_m) - z)^{-1}\varphi \big\rangle$$

holds for z with $|\mathrm{Im}\,z| \geq \sigma_n$ for some σ_n chosen such that $(\mathrm{d}\Gamma^{(n)}(\tilde{a}_m) - z)^{-1}$ preserves $\mathcal{D}(\mathrm{d}\Gamma^{(n)}(\omega))$ inside the n-particle sector, cf. Lemma A.8 3. By the unique continuation theorem, the identity then holds for all z with $\mathrm{Im}\,z \neq 0$. $\square$

Lemma 3.4. *Suppose* $\inf m(\omega) > 0$. *Then* $\widetilde{L}'_\beta$ *is of class* $C^1_{\mathrm{Mo}}(\widetilde{A})$, *with commutator* $[\widetilde{L}'_\beta, \widetilde{A}]^\circ = \imath\phi(\tilde{a}^2_m\widetilde{G}_\beta)$.

Proof. We aim to use Proposition A.9 to establish that $\widetilde{L}'_\beta$ is of class $C^1_{\mathrm{Mo}}(\widetilde{A})$. Note that $\widetilde{N}$ and $\widetilde{A}$ commute, and by Remark 2.8 both $\phi(\imath\tilde{a}_m\widetilde{G}_\beta)$ and $\phi(\tilde{a}^2_m\widetilde{G}_\beta)$ are $\sqrt{\widetilde{N}}$-bounded. Since $\mathcal{D}(\widetilde{L}_\beta) = \mathcal{D}(\widetilde{N})$ we conclude the first criteria, cf. Proposition A.9 2 2a. Since $\widetilde{\mathcal{C}}^{\mathrm{L}}$ is a core for both the commuting operators $\widetilde{N}$ and $\widetilde{A}$, it is dense in the intersection domain $\mathcal{D}(\widetilde{N}) \cap \mathcal{D}(\widetilde{A})$. Hence, to verify 2b and the desired form of the commutator, one simply has to verify the commutator identity in the sense of forms on $\widetilde{\mathcal{C}}^{\mathrm{L}}$. But this is straightforward. $\square$

We are now ready to state and prove our improvement of the Fröhlich-Merkli number bound. For comparison, we require one less commutator reflected in an improvement by one power of $|k|$ in the infrared behavior of G. It is however still one commutator more than what was needed for the Hamiltonian. It is unclear if this is just a technical issue.

Theorem 3.5. *Suppose* **(LG2)**. *Let* ψ *be an eigenstate of* L_β. *Then* $\psi \in \mathcal{D}(\sqrt{N^{\mathrm{L}}})$.

Proof. Let $\psi \in \mathcal{D}(\widetilde{L}_\beta)$ be an eigenstate for $\widetilde{L}_\beta$. It suffices to prove that $\psi \in \mathcal{D}(\widetilde{N}^{1/2})$. We can assume without loss of generality that the eigenvalue is zero, i.e. $\widetilde{L}_\beta \psi = 0$.

Denote by $\tilde{a} = \tilde{a}_{m \equiv 1}$ the generator of translations, cf. (3.46). Similarly we abbreviate $\widetilde{A} = \widetilde{A}_{m \equiv 1} = \mathbb{1}_{\mathcal{K} \otimes \mathcal{K}} \otimes d\Gamma(\tilde{a})$. Note that $\widetilde{A}$ commutes with $\widetilde{N}$.

Abbreviate $I_n(\widetilde{N}) = n(\widetilde{N} + 1)^{-1}$ as in Lemma A.11. Since $\widetilde{N}$ by Corollary 2.9 is of class $C^1_{\text{Mo}}(\widetilde{L}_\beta)$, we have $I_n(\widetilde{N})\psi \in \mathcal{D}(\widetilde{L}_\beta) \cap \mathcal{D}(\widetilde{N})$, for all $n \geq 1$. See Lemma A.5 2.

Put $\psi_n = I_n(\widetilde{N})\psi$. With the choice $m = 1$ we get

$$\widetilde{L}'_\beta = \widetilde{N} - \phi(\imath\tilde{a}\widetilde{G}_\beta),$$

as a self-adjoint operator with domain $\mathcal{D}(\widetilde{N})$. We can thus compute using Lemma 3.2 for $m \in \mathbb{N}$ and $z \in \mathbb{C}$, with $\text{Im}z \neq 0$,

$$\big\langle \psi_n, \imath[\widetilde{L}_\beta, (\widetilde{A}/m - z)^{-1}]\psi_n \big\rangle$$
$$= -\frac{1}{m}\big\langle \psi_n, (A/m - z)^{-1}\widetilde{L}'_\beta(\widetilde{A}/m - z)^{-1}\psi_n \big\rangle$$
$$= -\frac{1}{m}\big\langle \psi_n, \widetilde{L}'_\beta(\widetilde{A}/m - z)^{-2}\psi_n \big\rangle$$
$$\quad - \frac{1}{m^2}\big\langle \psi_n, (\widetilde{A}/m - z)^{-1}\phi(\tilde{a}^2\widetilde{G}_\beta)(\widetilde{A}/m - z)^{-2}\psi_n \big\rangle.$$

In the last equality we used Lemma 3.4. On the other hand we can undo the commutator on the left-hand side and commute $\widetilde{L}_\beta$ through $I_n(\widetilde{N})$ to get

$$\big\langle \psi_n, \imath[\widetilde{L}_\beta, (\widetilde{A}/m - z)^{-1}]\psi_n \big\rangle$$
$$= -\big\langle \psi_n, \big\{\phi(\imath\widetilde{G}_\beta)(\widetilde{N} + n)^{-1}(\widetilde{A}/m - z)^{-1} + (\widetilde{A}/m - z)^{-1}(\widetilde{N} + n)^{-1}\phi(\imath\widetilde{G}_\beta)\big\}\psi_n \big\rangle.$$

Here we used Corollary 2.9 and a twiddled version of (2.37).

Let $g \in C^\infty(\mathbb{R})$ be identical to t for $|t| \leq 1$, monotonously increasing and constant outside a ball of radius 2. Suppose in addition that $\sqrt{g'}$ is smooth. We will furthermore require that

$$\forall t \in \mathbb{R}: \quad tg''(t) \leq 0. \tag{3.49}$$

Let $\tilde{g}$ denote an almost analytic extension of g. Abbreviating $g_m(t) = mg(t/m)$ we get

$$-\big\langle \psi_n, \big\{\phi(\imath\widetilde{G}_\beta)(\widetilde{N} + n)^{-1}g_m(\widetilde{A}) + g_m(\widetilde{A})(\widetilde{N} + n)^{-1}\phi(\imath\widetilde{G}_\beta)\big\}\psi_n \big\rangle$$
$$= \big\langle \psi_n, \widetilde{L}'_\beta g'_m(\widetilde{A})\psi_n \big\rangle$$
$$\quad - \frac{1}{m\pi}\int_{\mathbb{C}} \bar{\partial}\tilde{g}(z)\big\langle \psi_n, (\widetilde{A}/m - z)^{-1}\phi(\tilde{a}^2\widetilde{G}_\beta)(\widetilde{A}/m - z)^{-2}\psi_n \big\rangle dz.$$

We estimate the left-hand side to be $O(m/\sqrt{n})$ and the second term on the right-hand side is $O(\sqrt{n}/m)$, cf. Remark 2.8. Hence we get

$$\big|\big\langle \psi_n, \widetilde{L}'_\beta g'_m(\widetilde{A})\psi_n \big\rangle\big| \leq C\Big(\frac{m}{\sqrt{n}} + \frac{\sqrt{n}}{m}\Big) \tag{3.50}$$

for some $C > 0$. Put $h(t) = \sqrt{g'(t)}$. Then $h \in C_0^\infty(\mathbb{R})$ and defining $h_m(t) = h(t/m)$ we find $h_m(\widetilde{A})^2 = g'_m(\widetilde{A})$. Let $\tilde{h}$ be an almost analytic extension of h. Then

$$\big\langle \psi_n, \widetilde{L}'_\beta g'_m(\widetilde{A})\psi_n \big\rangle = \big\langle \psi_n, h_m(\widetilde{A})\widetilde{L}'_\beta h_m(\widetilde{A})\psi_n \big\rangle + \big\langle \psi_n, [\widetilde{L}'_\beta, h_m(\widetilde{A})]h_m(\widetilde{A})\psi_n \big\rangle.$$

Observe that $[h_m(\widetilde{A}), \widetilde{L}'_\beta](\widetilde{N} + 1)^{-1/2} = [h_m(\widetilde{A}), \phi(\imath\widetilde{a}\widetilde{G}_\beta)](\widetilde{N} + 1)^{-1/2}$ is of the order $1/m$ and hence

$$\langle\psi_n, \widetilde{L}'_\beta g'_m(\widetilde{A})\psi_n\rangle = \langle\psi_n, h_m(\widetilde{A})\widetilde{L}'_\beta h_m(\widetilde{A})\psi_n\rangle + O\left(\frac{\sqrt{n}}{m}\right). \tag{3.51}$$

Finally using that $\widetilde{L}'_\beta \geq \widetilde{N}/2 - C'$, for some $C' > 0$, we get from (3.50) and (3.51) that

$$\langle\psi_n, h_m(\widetilde{A})\widetilde{N}h_m(\widetilde{A})\psi_n\rangle \leq C\left(\frac{m}{\sqrt{n}} + \frac{\sqrt{n}}{m}\right)$$

for some $C > 0$ and all $n, m \geq 1$.

We now pick $n = m^2$ such that we obtain the bound

$$\langle\psi, h_m(\widetilde{A})I_{m^2}(\widetilde{N})^2\widetilde{N}h_m(\widetilde{A})\psi\rangle \leq 2C$$

uniformly in m.

Let $E^{(\widetilde{N},\widetilde{A})}$ be the joint spectral resolution on $\mathbb{N}_0 \times \mathbb{R}$ induced by the two commuting operators $\widetilde{N}$ and $\widetilde{A}$. Then

$$\langle\psi, h_m(\widetilde{A})I_{m^2}(\widetilde{N})^2\widetilde{N}h_m(\widetilde{A})\psi\rangle = \int_{\mathbb{N}_0\times\mathbb{R}} h_m(t)^2\frac{nm^4}{(n+m^2)^2}\,\mathrm{d}E_\psi^{(\widetilde{N},\widetilde{A})}(n, t).$$

Since $h_m(t)^2\frac{nm^4}{(n+m^2)^2} \to n$ monotonously as $m \to \infty$ we conclude using the monotone convergence theorem that $\int_{\mathbb{N}_0\times\mathbb{R}} n\,\mathrm{d}E_\psi^{(\widetilde{N},\widetilde{A})}(n, t) < \infty$. Here we used (3.49) to ensure that $m \to h_m(t)$ is monotonously increasing towards 1. Being a joint spectral resolution we have

$$\int_{\mathbb{N}_0\times\mathbb{R}} n\,\mathrm{d}E_\psi^{(\widetilde{N},\widetilde{A})}(n, t) = \int_{\mathbb{N}_0} n\,\mathrm{d}E_\psi^{\widetilde{N}}(n),$$

where $E^{\widetilde{N}}$ is the spectral resolution for $\widetilde{N}$. Hence $\langle\psi, \widetilde{N}\psi\rangle < \infty$ and we are done. $\square$

3.3 Virial Theorems

Having established the number bounds we can now formulate and prove two virial theorems

Let $m \in C^1(\mathbb{R})$ be real-valued and bounded with bounded derivative as in the previous subsection. Given such a function m we can construct a maximally symmetric operator by the prescription

$$a_m = \frac{1}{2}\left\{\frac{m(|k|)k}{|k|} \cdot \nabla_k + \nabla_k \cdot \frac{m(|k|)k}{|k|}\right\}$$

on $C_0^\infty(\mathbb{R}^3\backslash\{0\})$. This gives rise to a maximally symmetric operator $A_m = \mathrm{d}\Gamma(a_m)$ on $\mathcal{H}$. We write, supposing (**HG1**),

$$H'_m = \mathrm{d}\Gamma(m(|k|)) - \phi(\imath a_m G),$$

as a form on $\mathcal{D}(\sqrt{N})$, cf. (2.22). If $\inf m(\omega) > 0$, then H' is self-adjoint on $\mathcal{D}(N)$. We have

Theorem 3.6. *Suppose* **(HG1)**. *Let* $\psi \in \mathcal{H}$ *be a bound state for the Hamiltonian* H. *Then* $\langle \psi, H'_m \psi \rangle = 0$.

Proof. Note that the expectation value is meaningful due to the number bound in Theorem 3.1.

Replace m by a regularizing function $m_n(r) = m(r)r/\sqrt{r^2 + n^{-1}}$ as in the proof of the number bound in Subsect. 3.1. Then the associated a_{m_n} is self-adjoint and so is $A_{m_n} = \mathrm{d}\Gamma(a_{m_n})$. Furthermore, H is of class $C^1(A_{m_n})$ for all n. We can compute the commutator $\imath[H, A_{m_n}]^\circ = \mathrm{d}\Gamma(m_n(|k|)) - \phi(\imath a_{m_n} G)$. By the usual virial theorem, cf. Theorem A.6, together with (3.45), Theorem 3.1 and Lebesgue's dominated convergence theorem we conclude the proof. $\qquad\square$

To deal with the standard Liouvillean we use the observables $\tilde{a}_m$ and $\widetilde{A}_m$ from (3.46) and (3.47). Write, supposing now **(LG1)**,

$$\widetilde{L}'_\beta = \mathbb{1}_{\mathcal{K}\otimes\mathcal{K}} \otimes \mathrm{d}\Gamma(m(\omega)) - \phi(\imath\tilde{a}_m\widetilde{G}_\beta),$$

which by Remark 2.8 is a well-defined form on $\mathcal{D}(\sqrt{\widetilde{N}})$. Again, if $\inf m(\omega) > 0$, then $\widetilde{L}'_\beta$ is self-adjoint on $\mathcal{D}(\widetilde{N})$. Define

$$L'_\beta = \mathcal{U}^* \widetilde{L}'_\beta \mathcal{U},$$

which is a well-defined form on $\mathcal{D}(\sqrt{N^{\mathrm{L}}})$. Under the **(LG1)** assumption we can compute

$$L'_\beta = \mathbb{1}_{\mathcal{K}\otimes\mathcal{K}} \otimes \big(\mathrm{d}\Gamma(m(|k|)) \otimes \mathbb{1}_\mathcal{F} + \mathbb{1}_\mathcal{F} \otimes \mathrm{d}\Gamma(m(|k|))\big) - \phi_\mathrm{l}(\imath a_m G_{\beta,\mathrm{l}}) + \phi_\mathrm{r}(\imath a_m G_{\beta,\mathrm{r}}).$$

We warn the reader that if one follows Remark 2.12 and imposes an **(LG1')** assumption instead of **(LG1)**, then $a_m G_{\beta,\mathrm{l/r}}$ may not be well-defined.

Theorem 3.7. *Suppose* **(LG2)**. *Let* $\psi \in \mathcal{H}^{\mathrm{L}}$ *be a bound state for the standard Liouvillean* L_β, *at inverse temperature* $0 < \beta \le \infty$. *Then* $\langle \psi, L'_\beta \psi \rangle = 0$.

Proof. First of all we note that the expectation value is meaningful due to Theorem 3.5. Secondly, it suffices to prove the theorem in the glued coordinates where L'_β is replaced by $\widetilde{L}'_\beta$ and ψ is an eigenstate for $\widetilde{L}_\beta$.

Let ψ be a bound state for $\widetilde{L}_\beta$. Using the notation $I_n(\widetilde{A}_m) = \imath n(\widetilde{A}_m + \imath n)^{-1}$, cf. Lemma A.11, we write

$$B_n = \widetilde{A}_m I_n(\widetilde{A}_m) = \imath n \mathbb{1}_{\widetilde{\mathcal{H}}^{\mathrm{L}}} + n^2(\widetilde{A}_m + \imath n)^{-1}.$$

Then B_n is bounded for all n. We compute using Lemma 3.2 as a form on $\mathcal{D}(\widetilde{L}_\beta) \cap \mathcal{D}(\sqrt{\widetilde{N}})$:

$$0 = \big\langle \psi, \imath[\widetilde{L}_\beta, B_n]\psi \big\rangle = \big\langle \psi, I_n(\widetilde{A}_m)\widetilde{L}'_\beta I_n(\widetilde{A}_m)\psi \big\rangle.$$

Since $\widetilde{N}$ commutes with $I_n(\widetilde{A}_m)$, we can - keeping Theorem 3.5 in mind - take the limit $n \to \infty$, using (A.89) and (A.91), and conclude the theorem. $\qquad\square$

The theorem of course remains true if we pass to the Jakšić-Pillet glued operator $\widetilde{L}_\beta$. While the proof given above is at least formally identical to a standard proof of the usual virial theorem, cf. the proof of Theorem A.6, the reader should keep in mind that it relies on the non-trivial Lemma 3.2 and Theorem 3.5.

The virial theorem's are the tools that will allow us to deduce statements about non-existence, local finiteness, and finite multiplicity, for eigenvalues given a so-called positive commutator estimate. This is the subject of Sect. 4

3.4 A Review of Existence and Non-existence Results

The first theorem we highlight is due to Gérard [29, Thm. 1] and establishes existence of a ground state for the Hamiltonian H under an **(HG1)** condition. Subsequently some improvements appeared in [12, 54].

Theorem 3.8. *Assume* **(HG1)**. *Then the bottom of the spectrum* Σ *of* H *is an eigenvalue.*

In a somewhat surprising recent development Hasler and Herbst proved that the Spin-Boson model, cf. Remark 2.12, admits a ground state if the coupling is sufficiently weak [35]. They used the renormalization group method of Bach, Fröhlich and Sigal [9]. See also Problems 3.3 and 3.4 in the following subsection.

The following beautiful theorem, due to Dereziński, Jakšić and Pillet establishes the existence of a β-KMS vector, which is in particular an eigenvector of L_β with eigenvalue zero. See [16, Thm. 7.3] and [17, Appendix B]. This improves on an earlier result of Bach, Fröhlich and Sigal [10, Thm. IV.3], who required more infrared regularity. For the particular case of the spin-boson model, the result goes back to [18].

Theorem 3.9. *Suppose* **(LG0)**. *Then for any inverse temperature* $0 < \beta < \infty$ *we have* $\Omega_\beta^{\mathrm{PF}} \in \mathcal{D}(\mathrm{e}^{-\beta(L_0 + \phi_{\beta,1}^{\mathrm{PF}}(G))})$ *and*

$$\Omega_{\beta,G}^{\mathrm{PF}} := \frac{\mathrm{e}^{-\beta(L_0 + \phi_{\beta,1}^{\mathrm{PF}}(G))}\Omega_\beta^{\mathrm{PF}}}{\|\mathrm{e}^{-\beta(L_0 + \phi_{\beta,1}^{\mathrm{PF}}(G))}\Omega_\beta^{\mathrm{PF}}\|} \in \mathcal{P}_\beta^{\mathrm{PF}} \cap \ker(L_\beta),$$

where $\mathcal{P}_\beta^{\mathrm{PF}}$ *is the standard cone* (1.18). *Furthermore,* $\Omega_{\beta,G}^{\mathrm{PF}}$ *is a faithful* β-KMS *vector for the standard Pauli-Fierz Liouvillean* L_β.

It is worth noting that although the above theorem mirrors Gérard's result for the Hamiltonian, it holds true for more singular interactions. In particular, one can not rule out a situation where H has no ground state, but L_β has a β-KMS vector in its kernel. Indeed, this situation actually occurs in the $\nu = 1$ case. Here the Pauli-Fierz Hamiltonian is of the type considered by Dereziński in [14], where it is referred to as a van Hove Hamiltonian. If we consider

$$G(k) = |k|^{-\frac{1}{2}}\hat{g}(k),$$

with $\hat{g} \in C_0^\infty(\mathbb{R}^3)$ real-valued playing the role of an ultraviolet cutoff. We put $\hat{g}(0) = 1$ such that the infrared behavior is captured by $|k|^{-1/2}$. It satisfies **(LG0)** needed for Theorem 3.9, but not **(HG1)** needed for Theorem 3.8.

With this coupling the Hamiltonian becomes of infrared type II, again referring to the terminology of [14], and does not admit a ground state. The ground state should be the coherent state $\mathrm{e}^{\mathrm{i}\phi(\mathrm{i}|k|^{-3/2}\hat{g})}|0\rangle$, but this is not in the Fock-space since $|k|^{-3/2}\hat{g} \notin L^2(\mathbb{R}^3)$. To see what happens with the standard "van Hove" Liouvillean we observe that for $\nu = 1$ (and real $\hat{g}$) we have

$$G_{\beta,\mathrm{l}} = G_{\beta,\mathrm{r}} = \left(\sqrt{1 + \rho_\beta} - \sqrt{\rho_\beta}\right)|k|^{-\frac{1}{2}}\hat{g}.$$

Expanding ρ_β around $k = 0$ we see that $\sqrt{1 + \rho_\beta} - \sqrt{\rho_\beta} \sim \sqrt{\beta|k|}/2$. Hence

$$G_{\beta,\mathrm{l/r}} \sim \frac{\sqrt{\beta}}{2} \tag{3.52}$$

at $k = 0$. Hence we can diagonalize the Liouvillean with a tensor product of Weyl operators as follows. Put

$$V = \mathrm{e}^{\imath\phi(\imath|k|^{-1}G_{\beta,\mathrm{l}})} \otimes \mathrm{e}^{\imath\phi(\imath|k|^{-1}G_{\beta,\mathrm{r}})},$$

which due to (3.52) is a well-defined unitary operator. Then $V^* L_\beta V = L_0$ and $V|0{\otimes}0\rangle$ is the only eigenstate and in particular the β-KMS state. Note that the energy shift one gets for the Hamiltonian does not occur here, since the shift from the left and right components cancel each other out.

Remark 3.10. In order to invoke the general results of [17], one must establish first that $\phi_{\beta,\mathrm{l}}(G)$ is a perturbation affiliated with the von Neumann algebra $\mathfrak{W}_{\beta,\mathrm{l}}$. The key is to observe that by the Trotter product formula [58, Thm. VIII.31]:

$$\mathrm{s} - \lim_{n\to\infty} \left(\mathrm{e}^{\imath\phi_{\beta,\mathrm{l}}(G_1)/n}\mathrm{e}^{\imath\phi_{\beta,\mathrm{l}}(G_2)/n}\right)^n = \mathrm{e}^{\imath\phi_{\beta,\mathrm{l}}(G_1+G_2)},$$

such that the set of G's for which $\mathrm{e}^{\imath\phi_{\beta,\mathrm{l}}(G)}$ is affiliated with $\mathfrak{W}_{\beta,\mathrm{l}}$ is a vector space over $\mathbb{C}$.

If G is of the form $G(k) = G_0 g(k)$, with $g \in L^2(\mathbb{R}^\nu)$ and $G_0 \in \mathrm{M}_\nu(\mathbb{C})$ self-adjoint, then one can find a basis $u_1, \ldots, u_\nu$ for $\mathbb{C}^\nu$, consisting of eigenvectors with $G_0 u_\ell = \lambda_\ell u_\ell$ and $\lambda_\ell \in \mathbb{R}$. Since for such G we have $\phi_{\beta,\mathrm{l}}(G) = \mathbf{s}^*(G_0 \otimes \mathbb{1}_\mathcal{K} \otimes \phi_{\beta,\mathrm{l}}^{\mathrm{AW}}(g))\mathbf{s}$, we get

$$\mathrm{e}^{\imath\phi_{\beta,\mathrm{l}}(G)} = \sum_{\ell=1}^\nu \mathbf{s}^*\big(|u_\ell\rangle\langle u_\ell| \otimes \mathbb{1}_\mathcal{K} \otimes W_{\beta,\mathrm{l}}^{\mathrm{AW}}(\lambda_\ell g)\big)\mathbf{s} \in \widetilde{\mathfrak{M}}_{\beta,\mathrm{l}} \subseteq \mathfrak{M}_{\beta,\mathrm{l}}.$$

Any $G_0 \in \mathrm{M}_\nu(\mathbb{C})$ can be written as a linear combination of self-adjoint matrices $G_0 = G_1 + \imath G_2$, and hence by the Trotter argument, $\phi_{\beta,\mathrm{l}}(G) = \phi_{\beta,\mathrm{l}}(G_1 g) + \phi_{\beta,\mathrm{l}}(G_2 \imath g)$, with $G(k) = G_0 g(k)$, is also affiliated with $\mathfrak{W}_{\beta,\mathrm{l}}$.

In order to deal with the general case, we take $G \in L^2(\mathbb{R}^\nu; \mathrm{M}_\nu(\mathbb{C}))$. Assume $\{G_n\} \subset L^2(\mathbb{R}^\nu; \mathrm{M}_\nu(\mathbb{C}))$ is a sequence converging to G in norm and such that $\phi_{\beta,\mathrm{l}}(G_n)$ is affiliated with $\mathfrak{M}_{\beta,\mathrm{l}}$. Such a sequence exist, since one can choose G_n to be simple functions, i.e. a finite linear combination of coupling functions of the form just handled above. Now one can show that on the analytic vectors $\psi \in \mathcal{C}^\mathrm{L}$, cf. (2.26), we have $\lim_{n\to\infty} \mathrm{e}^{\imath\phi_{\beta,\mathrm{l}}(G_n)}\psi = \mathrm{e}^{\imath\phi_{\beta,\mathrm{l}}(G)}\psi$. Hence by density we get strong convergence and this concludes the proof of the perturbation $\phi_{\beta,\mathrm{l}}(G)$ being affiliated with $\mathfrak{W}_{\beta,\mathrm{l}}$. ◇

Theorem 3.9 is also the missing ingredient in the

Proof of Theorem 2.11. It suffices to prove the theorem for $\widetilde{L}_\beta$ and show that $\sigma((\widetilde{L}_\beta + \imath)^{-1}) = (\mathbb{R}+\imath)^{-1}$. Fix $\lambda \in \mathbb{R}\backslash\{0\}$ and put $\psi_m = m(\widetilde{N}+m)^{-1}\Omega_{\beta,G}^{\mathrm{PF}}$.

Since $\widetilde{N}$ is of class $C_\mathrm{Mo}^1(\widetilde{L}_\beta)$, cf. Corollary 2.9 1, we find that $\psi_m \in \mathcal{D}(\widetilde{N}) \cap \mathcal{D}(\widetilde{L}_\beta)$. Furthermore,

$$\widetilde{L}_\beta\psi_m = m\widetilde{L}_\beta(\widetilde{N}+m)^{-1}\Omega_{\beta,G}^{\mathrm{PF}} = (\widetilde{N}+m)^{-1}\tilde{\phi}(\imath\widetilde{G}_\beta)\psi_m.$$

Since the right-hand side is an element of $\mathcal{D}(\widetilde{N})$ we can conclude that $\widetilde{L}_\beta\psi_m \in \mathcal{D}(\widetilde{N})$ and

$$(\widetilde{N}+1)^{\frac{1}{2}}\widetilde{L}_\beta\psi_m = (\widetilde{N}+1)^{\frac{1}{2}}(\widetilde{N}+m)^{-1}\tilde{\phi}(\imath\widetilde{G}_\beta)\psi_m.$$

We thus get

$$\big\|(\widetilde{N}+1)^{\frac{1}{2}}\widetilde{L}_\beta\psi_m\big\| \le C\big\|m^{\frac{1}{2}}(\widetilde{N}+1)^{\frac{1}{2}}(\widetilde{N}+m)^{-1}\Omega_{\beta,G}^{\mathrm{PF}}\big\|$$
$$\le C\big\|m(N+m)^{-1}\Omega_{\beta,G}^{\mathrm{PF}}\big\|^{\frac{1}{2}}\big\|(N+1)(N+m)^{-1}\Omega_{\beta,G}^{\mathrm{PF}}\big\|^{\frac{1}{2}},$$

where the last factor on the right-hand side goes to zero in the large m limit. Hence we can pick m large enough such that with $\psi = \psi_m$ we have

$$\big\|(\widetilde{N}+1)^{\frac{1}{2}}\widetilde{L}_\beta\psi\big\| \le \epsilon. \tag{3.53}$$

Choose $h_n \in C_0^\infty(\mathbb{R})$ real-valued with $\|h\| = 1$ (L^2-norm) and $\mathrm{supp}(h) \subseteq [-1,1]$. Put $h_n(\omega,\Theta) = h_n(\omega) = n^{1/2}h(n(\omega-\lambda))$. Define $\psi_n = (a^*(h_n)\psi$. Using now the pullthrough formula from Proposition 2.10, cf. the proof of Theorem 2.5, we get for $\varphi \in \mathcal{D}(\widetilde{N})$

$$\big\langle\varphi, \big((\widetilde{L}_\beta+\imath)^{-1} - (\lambda+\imath)^{-1}\big)\psi_n\big\rangle$$
$$= \int_{\mathbb{R}\times S^2} h_n(\omega)\big\langle\big((L_\beta+\omega-\imath)^{-1} - (\lambda-\imath)^{-1}\big)a(\omega,\Theta)\varphi,\psi\big\rangle\,\mathrm{d}\omega\mathrm{d}\Theta$$
$$- \int_{\mathbb{R}\times S^2} \frac{h_n(\omega)}{\sqrt{2}}\big\langle\widetilde{G}_\beta(\omega,\Theta)\otimes\mathbb{1}_{\widetilde{\mathcal{F}}}\varphi, (\widetilde{L}_\beta+\omega+\imath)^{-1}\psi\big\rangle\,\mathrm{d}\omega\mathrm{d}\Theta.$$

Inserting $(\widetilde{N}+2)^{1/2}(\widetilde{N}+2)^{1/2}$ to handle the annihilation operator, we estimate the first integrand on the right-hand side

$$\big|h_n(\omega)\big\langle\big((\widetilde{L}_\beta+\omega-\imath)^{-1} - (\lambda-\imath)^{-1}\big)a(k)\varphi,\psi\big\rangle\big|$$
$$\le |h_n(\omega)|\big\|(\widetilde{N}+2)^{-\frac{1}{2}}a(k)\varphi\big\|$$
$$\times \big\|(\widetilde{N}+2)^{\frac{1}{2}}(\widetilde{L}_\beta+\omega+\imath)^{-1}\big(\widetilde{L}_\beta+(\omega-\lambda)\big)\psi\big\|$$
$$\le C|h_n(\omega)|\big\|(\widetilde{N}+1)^{\frac{1}{2}}\big(\widetilde{L}_\beta+(\omega-\lambda)\big)\psi\big\|\big\|a(k)(\widetilde{N}+1)^{-1}\varphi\big\|$$
$$\le C|h_n(\omega)|\Big(\epsilon+\frac{1}{n}\Big)\big\|a(k)(\widetilde{N}+1)^{-1}\varphi\big\|.$$

Here we used (3.53) in the last step, and in the second step we employed Proposition A.13 2 to move the root of the number operator through $(\widetilde{L}_\beta+\omega+\imath)^{-1}$. We now conclude the bound

$$\big|\big\langle\varphi, \big((\widetilde{L}_\beta+\imath)^{-1} - (\lambda+\imath)^{-1}\big)\psi_n\big\rangle\big| \le C(\epsilon+o(1))\|\varphi\|.$$

As in the proof of Theorem 2.5, it remains to observe that $\|\psi_n\|$ is bounded from below uniformly in n. $\qquad\square$

The final result we discuss in this subsection is a consequence of Theorem 3.9 and a theorem of Jadczyk [38], which has as a consequence that existence and simplicity of the 0 eigenvalue for the standard Liouvillean implies non-existence of non-zero eigenvalues! We refer the reader to the short and very elegant paper [40] for details, which are entirely operator algebraic in nature.

Theorem 3.11. *Suppose* **(LG0)**. *Let* $0 < \beta < \infty$ *and suppose that* 0 *is a simple eigenvalue for* L_β. *Then* $\sigma_{\mathrm{pp}}(L_\beta) = \{0\}$.

3.5 Open Problems III

As the reader may have observed, the bottleneck for applying the virial theorem to the standard Liouvillean is the number bound Theorem 3.5, where we compared with the Hamiltonian case Theorem 3.1 need much stronger assumptions. This is in particular unfortunate since the positive commutator estimates we establish in the following section hold under an **(LG1)** assumption, not the **(LG2)** assumption needed for the number bound.

Problem 3.1. Can the number bound in Theorem 3.5 be established under an **(LG1)** condition, or some other condition truly weaker than **(LG2)**.

The author does not know one way or the other what the answer may be to this problem. We remark that although, the number bound is a bottleneck viz a viz the structure of the point spectrum, the **(LG2)** condition is what one would expect for a limiting absorption principle to hold, given a positive commutator estimate. Hence, from a broader perspective the **(LG2)** condition will appear anyway.

The proof of the number bound Theorem 3.5 did not make essential use of the small system being finite dimensional. Hence we expect the theorem to remain true also for confined small systems, like the standard Liouvillean for the confined Nelson model.

Problem 3.2. Extend Theorem 3.5 to the case where the small system $\mathcal{K}$ is not necessarily finite dimensional.

As mentioned in Subsect. 3.4, Hasler and Herbst established in [35] the existence of an interacting ground state for the spin-boson model with physical infrared singularity $|k|^{-1/2}$, provided the coupling is sufficiently weak. This result came as a complete surprise to the author, since it is contrary to the solvable model with $\mathcal{K} = \mathbb{C}$ and the confined Nelson model [12, 14, 37, 46, 55]. Furthermore, it goes beyond what was considered the natural borderline established in [29], cf. also [7, 12, 54]. In fact there has been speculation that gauge invariance of the minimally coupled model was responsible for the existence result of Griesemer-Lieb-Loss [33, 44], something that was however debunked by Hasler-Herbst [34, 36] who proved that existence of a ground state, at weak coupling, remains true even after dropping the quadratic term in the minimally coupled model, thus breaking gauge invariance.

The $|k|^{-1/2}$ infrared behavior of G is sometimes called the "ohmic case", a terminology we use below.

Problem 3.3. Does there exists a critical coupling at which the ground state seize to exist for the spin-boson model considered by Hasler and Herbst? Or does a ground state exist for all couplings?

Problem 3.4. Characterize the properties of ohmic G that ensures existence of a ground state for H in the weak coupling regime. As a simpler problem, consider G's on the form $G(k) = |k|^{-1/2}\hat{g}(k)G_0$ as discussed in Remark 2.12.

For the thermal standard Liouvillean, one has existence of a β-KMS vector in the kernel of L_β at all values of β, cf. Theorem 3.9, and furthermore the modular structure ensures that a simple 0-eigenvalue implies absence of non-zero eigenvalues, cf. Theorem 3.11. These results were derived from the underlying algebraic structure of standard Liouvilleans, and may not have natural operator theoretic proofs. It would be natural to ask if it is not possible to extract even more information from the underlying algebraic framework.

Problem 3.5. Can one exploit the underlying algebraic structure to infer more information on the point spectrum and pertaining eigenstates, than what is afforded by The-

orem 3.11. In particular, can one use algebraic arguments to conclude that zero is in fact a simple eigenvalue of L_β?

It is well known that establishing instability or outright absence of embedded eigenvalues away from zero coupling, or some other explicitly solvable regime, is a daunting task. It is for example not known if embedded (necessarily negative) eigenvalues of N-body Schrödinger operators are unstable under perturbations of pair-potentials. One can only show generic instability under perturbations by external potentials cf. [1, 2]. In [20] a Fermi Golden Rule was established at arbitrary coupling for the Hamiltonian, but to conclude instability one needs better control of eigenstates beyond the ground state (where Perron-Frobenius theory applies). The case of perturbation around zero coupling is far better understood [9, 10, 11, 15, 16, 23, 32, 47]. Hence, whether or not the kernel of the standard Liouvillean is generically one-dimensional beyond the weak-coupling regime is not a question one is likely to answer using perturbation theory of embedded eigenvalues only.

There is perhaps an unexplored avenue available for investigating Problem 3.5, which we now discuss. Let $T = \mathrm{e}^{-L_\beta^2/2}$. Then T is a bounded self-adjoint operator with an eigenvalue sitting at the top of its spectrum. Appealing to the Fourier transform we observe that

$$T = (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} \mathrm{e}^{-s^2/2} \mathrm{e}^{\imath s L_\beta} \, \mathrm{d}s.$$

Since L_β is a standard Liouvillean, $\mathrm{e}^{\imath s L_\beta}$ preserves the self-dual standard cone $\mathcal{P}_\beta^{\mathrm{PF}}$, cf. (1.18). Hence T is positivity preserving and it thus seems natural to apply Perron-Frobenius theory in an abstract form. There is however an obstacle to this approach, namely the fact that the order on $\mathcal{H}^{\mathrm{L}}$ induced by the standard cone is not a lattice. Hence, one can not map the problem into a function space, where the standard cone goes into the positive functions. See [56]. If that had been possible one could have made use of ergodicity arguments as in [35].

We stress that we consider Problem 3.5 to be the most important problem highlighted in these notes. The reason being that, due to Theorem 3.11, it reduces the question of establishing mixing a.k.a. "return to equilibrium" beyond the weak coupling regime to positive commutator estimates and limiting absorption principles. Something we see no fundamental obstacle to obtaining, although the picture is not yet entirely clear beyond small temperatures. See Sect. 5.

Finally, it would be natural, in the spirit of [14], to investigate the types of ultraviolet and infrared behavior of the standard Liouvillean when $\nu = 1$, which is a solvable case. See also the discussion on ground states versus β-KMS states when $\nu = 1$ in the previous subsection, which indicates that the infrared type II property, cf. [14], characterizes existence of β-KMS states.

Problem 3.6. Classify possible types of ultraviolet and infrared behavior of the "van Hove Liouvillean", i.e. when $\nu = 1$.

4 Commutator Estimates

4.1 The Weak Coupling Regime

The weak coupling regime is very well understood. To explore it we replace G by λG, where $\lambda \in \mathbb{R}$ is small in norm. In fact, obtaining positive commutator estimates in this regime is an easy exercise. Indeed, choosing a to be generator of radial translation (3.44) we get using (2.22)

$$H' = N - \lambda\phi(\imath a G) \geq \frac{1}{2}N - \lambda^2\|aG\|^2.$$

Choosing λ such that $\lambda^2\|aG\|^2 \leq 1/4$ yields

$$H' \geq \frac{1}{4}\mathbb{1}_{\mathcal{H}} - \frac{1}{4}\mathbb{1}_{\mathcal{K}} \otimes |0\rangle\langle 0|. \tag{4.54}$$

We can now prove

Corollary 4.1. *Suppose* **(HG1)** *and let* $\lambda_0 = \|aG\|^{-1}$. *For* $\lambda \in (-\lambda_0, \lambda_0)$ *the pure point spectrum* $\sigma_{\mathrm{pp}}(H)$ *is finite and all eigenvalues have finite multiplicity. Indeed,* $\dim\mathrm{Ran}(P) \leq \nu$, *where* $P = E(\sigma_{\mathrm{pp}}(H))$ *is the projection onto the pure point subspace. (Here H is defined with G replaced by λG.)*

Proof. Let $\{\psi_j\}_{j=1}^n$ be an orthonormal set of eigenvectors for H. We can use the virial theorem, Theorem 3.6, together with (4.54) to estimate

$$0 = \sum_{j=1}^n \langle\psi_j, H'\psi_j\rangle \geq \frac{n}{4} - \frac{\mathrm{Tr}(\mathbb{1}_{\mathcal{K}} \otimes |0\rangle\langle 0|)}{4} = \frac{n-\nu}{4}.$$

This implies $n \leq \nu$ and concludes the proof. $\qquad\square$

Similarly for the Liouvillean where we can again choose $\tilde{a} = \imath\frac{\mathrm{d}}{\mathrm{d}\omega} \otimes \mathbb{1}_{L^2(S^2)}$ to be the generator of translations in the glued variable. Then

$$\widetilde{L}'_\beta = \widetilde{N}^{\mathrm{L}} - \lambda\phi(\imath\tilde{a}\widetilde{G}_\beta) \geq \frac{1}{2}\widetilde{N}^{\mathrm{L}} - \lambda^2\big\|\tilde{a}\widetilde{G}_\beta\big\|^2.$$

Hence we arrive at

Corollary 4.2. *Suppose* **(LG2)** *and let* $\lambda_0 = \|\tilde{a}\widetilde{G}\|^{-1}$. *For* $\lambda \in (-\lambda_0, \lambda_0)$ *the pure point spectrum* $\sigma_{\mathrm{pp}}(L_\beta)$ *is finite and all eigenvalues have finite multiplicity. Indeed,* $\dim\mathrm{Ran}(P_\beta) \leq \nu^2$, *where* $P_\beta = E^\beta(\sigma_{\mathrm{pp}}(L_\beta))$ *is the projection onto the pure point subspace. (Here L_β is defined with G replaced by λG.)*

Proof. The proof is identical to the proof of Corollary 4.1, except we make use of Theorem 3.7 instead of Theorem 3.6. $\qquad\square$

This theorem improves on a result of Merkli [47] due to the improvement in the number bound Theorem 3.5. See also [23, 24, 25].

4.2 Conjugate Operators

Let $\chi \in C_0^\infty(\mathbb{R})$ satisfy $\chi(-\omega) = \chi(\omega)$, $0 \leq \chi \leq 1$, $\chi(\omega) = 1$ for $|\omega| \leq 1/2$ and $\chi(\omega) = 0$ for $|\omega| > 1$.

Let $\mu > 0$ be the constant used to define the class of couplings we can treat, cf. **(HGn)** and **(LGn)**. We use it to construct an auxiliary function $d \colon (0, \infty) \to [1, \infty)$ as follows

$$d(\omega) = \chi(\omega)\omega^{-\mu/4} + \chi(\omega/2) - \chi(\omega) + (1 - \chi(\omega/2))\omega^{\mu/4}.$$

We leave it to the reader to verify the following properties of d

(d1) $(\omega - 1)d'(\omega) \geq 0$.

(d2) $\lim_{\omega \to 0_+} d(\omega) = \lim_{\omega \to +\infty} d(\omega) = +\infty$.

(d3) $\exists C > 0$ s.t. $|d'(\omega)| \leq C d(\omega)/\omega$ for all $\omega > 0$.

We extend d to $\mathbb{R} \setminus \{0\}$ be setting $d(\omega) = d(-\omega)$ for $\omega < 0$.

For a given

$$\underline{\delta} = (\delta_0, \delta_\infty) \in (0, 1] \times [1, \infty) =: \Delta_0 \tag{4.55}$$

we define a smooth positive function $m_\delta \colon \mathbb{R} \to [1, \infty)$ by

$$
\begin{aligned}
m_{\underline{\delta}}(\omega) = {}& d(\delta_0)\chi(\omega/\delta_0) \\
& + d(\omega)\big(\chi(\omega/(2\delta_\infty)) - \chi(\omega/\delta_0)\big) \\
& + d(\delta_\infty)\big(1 - \chi(\omega/(2\delta_\infty))\big).
\end{aligned}
$$

Observe that $m_{\underline{\delta}}$ has bounded and compactly supported derivatives.

Our conjugate operator on the one-particle level for the Hamiltonian at zero temperature is defined as the modified generator of radial translations

$$a_{\underline{\delta}} = \frac{1}{2}\left\{ m_{\underline{\delta}}(k)\frac{k}{|k|} \cdot \nabla_k + \nabla_k \cdot \frac{k}{|k|}m_{\underline{\delta}}(k) \right\}.$$

Note that $a_{\underline{\delta}}$ a priori defined on $C_0^\infty(\mathbb{R}^3 \setminus \{0\})$ is closable and its closure is a maximally symmetric operator. The conjugate operator is obtained through second quantization

$$A_{\underline{\delta}} = \mathbb{1}_{\mathcal{K}} \otimes \mathrm{d}\Gamma(a_{\underline{\delta}})$$

and is again a maximally symmetric operator closable on $\mathcal{K} \otimes \Gamma_{\mathrm{fin}}(C_0^\infty(\mathbb{R}^3 \setminus \{0\}))$.

To get a conjugate operator for the Liouvillean we do the construction after gluing and define the modified generator of translations

$$\tilde{a}_{\underline{\delta}} := \frac{1}{2}\left\{ m_{\underline{\delta}}(\omega)\frac{\mathrm{d}}{\mathrm{d}\omega} + \frac{\mathrm{d}}{\mathrm{d}\omega}m_{\underline{\delta}}(\omega) \right\} \otimes \mathbb{1}_{L^2(S^2)},$$

which is essentially self-adjoint on $C_0^\infty(\mathbb{R}) \otimes C^\infty(S^2)$. We second quantize to obtain

$$\widetilde{A}_{\underline{\delta}} := \mathbb{1}_{\mathcal{K} \otimes \mathcal{K}} \otimes \mathrm{d}\Gamma(\tilde{a}_{\underline{\delta}}),$$

which is essentially self-adjoint on $\widetilde{\mathcal{C}}^{\mathrm{L}}$. Note that we have simplified the notation a bit writing $\tilde{a}_{\underline{\delta}}$ and $\widetilde{A}_{\underline{\delta}}$ instead of the more cumbersome $\tilde{a}_{m_{\underline{\delta}}}$ and $\widetilde{A}_{m_{\underline{\delta}}}$, cf. (3.46) and (3.47).

As an identity on $C_0^\infty(\mathbb{R} \setminus \{0\}) \otimes C^\infty(S^2)$ we have $T(a_{\underline{\delta}} \otimes \mathbb{1}_\mathfrak{h} - \mathbb{1}_\mathfrak{h} \otimes a_{\underline{\delta}})T^* = \tilde{a}_{\underline{\delta}}$ and furthermore

$$\mathcal{U}\widetilde{L}'_\infty \mathcal{U}^* = L'_\infty = H' \otimes \mathbb{1}_\mathcal{H} + \mathbb{1}_\mathcal{H} \otimes H^{\mathbf{c}'}. \tag{4.56}$$

The latter being an operator identity on $\mathcal{D}(N^{\mathrm{L}})$.

4.3 Estimates at Zero Temperature

Throughout this section we will for $\underline{\delta}' \in \Delta_0$, cf. (4.55), use the notation

$$\Delta(\underline{\delta}') := \left\{\underline{\delta} \in \Delta_0 \,\middle|\, \delta_0 \leq \delta_0', \delta_\infty \geq \delta_\infty'\right\}.$$

We begin with a new high-energy estimate, which is particular to the case of finite dimensional small systems. It will not hold e.g. for (confined) atomic small systems.

For $\underline{\delta} \in \Delta_0$, we write $N_{\underline{\delta}}$ for $d\Gamma(m_{\underline{\delta}})$, the modified number operator appearing in $H' = N_{\underline{\delta}} - \phi(\imath a_{\underline{\delta}} G)$. The reason for introducing the modified generator of radial translation is that $N_{\underline{\delta}}$ is large in the infrared and ultraviolet regimes, which allows us to handle very soft and very hard photons.

Theorem 4.3. *Suppose* **(HG1)**. *Let $e > 0$ be given. There exists $\delta_\infty' \geq 1$, $c > 0$ and $E_0 \in \mathbb{R}$ such that for all $\underline{\delta} \in \Delta((1, \delta_\infty'))$ we have*

$$H' \geq e\mathbb{1}_{\mathcal{H}} - c\mathbb{1}[H \leq E_0]$$

in the sense of forms on $\mathcal{D}(N)$.

Proof. The first step we take is to estimate from below

$$H' \geq \frac{1}{2}N_{\underline{\delta}} - C\mathbb{1}_{\mathcal{H}} \tag{4.57}$$

exploiting the $N^{1/2}$-boundedness of $\phi(\imath a_{\underline{\delta}} G)$, cf. (2.22), and the inequality $N \leq N_{\underline{\delta}}$. Here C is some positive number.

For $R > 1$ we perform a partition of unity in momentum space as follows. Let

$$F^R = \begin{pmatrix} \mathbb{1}[|k| < R] \\ \mathbb{1}[|k| \geq R] \end{pmatrix} : \mathfrak{h} \to L^2(B(0,R)) \oplus L^2(B(0,R)^c) =: \mathfrak{h}_<^R \oplus \mathfrak{h}_>^R \tag{4.58}$$

and observe that F^R is unitary. We lift to $\mathcal{F}$ and get

$$\check{\Gamma}(F^R) = I\Gamma(F^R) : \mathcal{F} \to \Gamma(\mathfrak{h}_<^R) \otimes \Gamma(\mathfrak{h}_>^R) =: \mathcal{F}_<^R \otimes \mathcal{F}_>^R. \tag{4.59}$$

Put $H_0^\times = K \otimes \mathbb{1} + \mathbb{1}_{\mathcal{K}} \otimes H_{\mathrm{ph}|\mathcal{F}_<^R} \otimes \mathbb{1} + \mathbb{1} \otimes H_{\mathrm{ph}|\mathcal{F}_>^R}$ and abbreviate $\lambda_{\max} = \max \sigma(K)$. We now compute for $\widetilde{E} > \lambda_{\max}$

$$N_{\underline{\delta}} \geq N_{\underline{\delta}}\mathbb{1}[H_0 > \widetilde{E}]$$
$$= \check{\Gamma}(F^R)^* \left\{ \mathbb{1}_{\mathcal{K}} \otimes N_{\underline{\delta}|\mathcal{F}_<^R} \otimes \mathbb{1}_{\mathcal{F}_>^R} + \mathbb{1}_{\mathcal{K} \otimes \mathcal{F}_<^R} \otimes N_{\underline{\delta}|\mathcal{F}_>^R} \right\} \mathbb{1}[H_0^\times > \widetilde{E}]\check{\Gamma}(F^R)$$
$$\geq \Gamma(\mathbb{1}[|k| < R])N_{\underline{\delta}}\mathbb{1}[H_0 > \widetilde{E}] + m_\delta(R)\check{\Gamma}(F^R)^*\mathbb{1}_{\mathcal{K} \otimes \mathcal{F}_<^R} \otimes \overline{P}_\Omega \mathbb{1}[H_0^\times > \widetilde{E}]\check{\Gamma}(F^R)$$
$$\geq \frac{\widetilde{E} - \lambda_{\max}}{R}\Gamma(\mathbb{1}[|k| < R])\mathbb{1}[H_0 > \widetilde{E}] + m_{\underline{\delta}}(R)\check{\Gamma}(R)^*\mathbb{1}_{\mathcal{K} \otimes \mathcal{F}_<^R} \otimes \overline{P}_\Omega \mathbb{1}[H_0^\times > \widetilde{E}]\check{\Gamma}(F^R)$$
$$\geq \min\left\{ \frac{\widetilde{E} - \lambda_{\max}}{R}, m_{\underline{\delta}}(R) \right\} \mathbb{1}[H_0 > \widetilde{E}].$$

We thus get

$$\frac{1}{2}N_{\underline{\delta}} - C\mathbb{1}_{\mathcal{H}} \geq \frac{1}{2}\min\left\{ \frac{\widetilde{E} - \lambda_{\max}}{R}, m_{\underline{\delta}}(R) \right\} \mathbb{1}[H_0 > \widetilde{E}] - C\mathbb{1}_{\mathcal{H}}. \tag{4.60}$$

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To pass from H_0 to H we estimate, recalling that Σ denotes the bottom of the spectrum of H (2.25),

$$
\begin{aligned}
\mathbb{1}[H_0 \leq \widetilde{E}] &\leq (\widetilde{E}+1)(H_0+1)^{-1} \\
&= (\widetilde{E}+1)(H-\Sigma+1)^{-\frac{1}{2}} \\
&\quad \times \left\{(H-\Sigma+1)^{\frac{1}{2}}(H_0+1)^{-1}(H-\Sigma+1)^{\frac{1}{2}}\right\}(H-\Sigma+1)^{-\frac{1}{2}} \\
&\leq (\widetilde{E}+1)\widetilde{C}(H-\Sigma+1)^{-1} \\
&\leq (\widetilde{E}+1)\widetilde{C}\mathbb{1}[H \leq E] + \frac{(\widetilde{E}+1)\widetilde{C}}{E-\Sigma+1}.
\end{aligned}
$$

Combining with (4.57) and (4.60) we arrive at the bound

$$
H' \geq \frac{1}{2}\min\left\{\frac{\widetilde{E}-\lambda_{\max}}{R}, m_{\underline{\delta}}(R)\right\}\left(1 - \frac{\widetilde{C}(\widetilde{E}+1)}{E-\Sigma+1}\right) - C - \frac{(\widetilde{E}+1)\widetilde{C}}{2}\mathbb{1}[H \leq E].
$$

We are now in a position to pick our constants. First choose R large enough such that $d(R)/2 \geq e+C+1$. Then choose $\delta'_\infty \geq R$ such that for $\underline{\delta} \in (0,1) \times (\delta'_\infty, \infty)$ we have $m_{\underline{\delta}}(R) = d(R)$. Subsequently we fix $\widetilde{E}$ large enough such that $(\widetilde{E}-\lambda_{\max})/(2R) \geq e+C+1$. With these choices of R, δ' and $\widetilde{E}$ (in that order) we get

$$
H' \geq (e+C+1)\left(1 - \frac{\widetilde{C}(\widetilde{E}+1)}{E-\Sigma+1}\right) - C - \frac{(\widetilde{E}+1)\widetilde{C}}{2}\mathbb{1}[H \leq E].
$$

Finally we can take E_0 large enough such that with $E = E_0$ the right-hand side is bounded from below by $e\mathbb{1}_{\mathcal{H}} - \frac{1}{2}\widetilde{C}(\widetilde{E}+1)\mathbb{1}[H \leq E_0]$. $\qquad\square$

For the purpose of the following we introduce the terminology that H satisfies a Mourre estimate at $E \in \mathbb{R}$:

Definition 4.4. We say that H satisfies a Mourre estimate at $E \in \mathbb{R}$ if there exists $\underline{\delta}' \in \Delta_0$ such that: For all $\epsilon > 0$ there exist $C > 0$, $\kappa > 0$ and a compact operator K such that as a form on $\mathcal{D}(N)$

$$
H' \geq (1-\epsilon)\mathbb{1} - C\mathbb{1}[|H-E| > \kappa] - K,
$$

for any $\underline{\delta} \in \Delta(\underline{\delta}')$.

Theorem 4.3 implies that a Mourre estimate is satisfied at any $E > E_0$. Since H' is bounded from below, we also get a Mourre estimate automatically satisfied at any $E < \Sigma$. Note that the Mourre estimate obviously remains true if we replace κ by any smaller positive κ'.

Lemma 4.5. *Suppose* **(HG1)**. *Let $J \subset \mathbb{R}$ be a compact set with a Mourre estimate satisfied at all $E \in J$. Then there exists $\underline{\delta}' \in \Delta((1,1))$ such that: For any $\epsilon > 0$, there exist $\kappa > 0$, $C > 0$, such that for all $E \in J$ and $\underline{\delta} \in \Delta(\underline{\delta}')$ we have*

$$
H' \geq -\epsilon\mathbb{1}_{\mathcal{H}} - C\mathbb{1}[|H-E| \geq \kappa],
$$

in the sense of forms on $\mathcal{D}(N)$.

Remark 4.6. If a Mourre estimate holds at all $E \in J$ with the same $\underline{\delta}'$, then this $\underline{\delta}'$ can also be used for the uniform bound. This will be evident from the proof below. $\diamond$

Proof. First note that by the virial theorem, the point spectrum in an open neighborhood of J is locally finite and eigenvalues in J have finite multiplicity.

We begin by verifying the estimate for a fixed E, for which the Mourre estimate is satisfied. If $E \notin \sigma_{\mathrm{pp}}(H)$ we proceed as follows: First extract a Mourre estimate with the given ϵ. Write for $0 < \kappa' < \kappa$ the compact error as $K = K\mathbb{1}[|H - E| \leq \kappa'] + K\mathbb{1}[|H - E| > \kappa']$. Pick κ' small enough such that $\|K\mathbb{1}[|H - E| \leq \kappa']\| \leq 1/2$. Then

$$H' \geq -\epsilon\mathbb{1} - (C + \|K\|)\mathbb{1}[|H - E| > \kappa'].$$

If on the other hand $E \in \sigma_{\mathrm{pp}}(H)$ we proceed differently. Write P_E for the finite rank orthogonal projection on the eigenspace associated with E. Abbreviate $\overline{P}_E = \mathbb{1} - P_E$. Since $\mathrm{Ran}(P_E) \subset \mathcal{D}(N^{1/2})$ we can compute and estimate

$$
\begin{aligned}
H' &= P_E H' P_E + 2\mathrm{Re}\{P_E H' \overline{P}_E\} + \overline{P}_E H' \overline{P}_E \\
&= 2\mathrm{Re}\{P_E H' \overline{P}_E\} + \overline{P}_E H' \overline{P}_E \\
&\geq (1 - \epsilon/5)\overline{P}_E - C\mathbb{1}[|H - E| \geq \kappa] - \overline{P}_E K \overline{P}_E + 2\mathrm{Re}\{P_E H' \overline{P}_E\} \\
&\geq -\epsilon/5\mathbb{1} - C\mathbb{1}[|H - E| \geq \kappa'] + 2\mathrm{Re}\{P_E H' \overline{P}_E\}, \quad\quad (4.61)
\end{aligned}
$$

where we used Theorem 3.6 in the second equality and for the first inequality we used the assumed to hold Mourre estimate (applied with ϵ replaced by $\epsilon/5$). In the last step we argued as above to get rid of the compact error by passing to a smaller $\kappa' < \kappa$.

As for the cross term (4.61) we write $H' = N_{\underline{\delta}} - \phi(\imath a_{\underline{\delta}} G)$ as a form sum on $D(N^{1/2})$. Recalling Theorem 3.1, we decompose for an $r > 0$ to be fixed later

$$P_E H' \overline{P}_E = P_E N_{\underline{\delta}} \mathbb{1}[N_{\underline{\delta}} > r]\overline{P}_E + K\overline{P}_E,$$

with $K = P_E N_{\underline{\delta}} \mathbb{1}[N_{\underline{\delta}} \leq r] - P_E \phi(\imath a_{\underline{\delta}} G)$ being compact. Estimate first for $\sigma > 0$

$$
\begin{aligned}
2\mathrm{Re}\{P_E N_{\underline{\delta}} \mathbb{1}[N_{\underline{\delta}} > r]\overline{P}_E\} &= 2\mathrm{Re}\{P_E N_{\underline{\delta}} \mathbb{1}[N_\delta > r]\} - 2P_E N_{\underline{\delta}} \mathbb{1}_{N_{\underline{\delta}} > r} P_E \\
&\geq -\sigma N_{\underline{\delta}} - (2 + \sigma^{-1}) P_E N_{\underline{\delta}} \mathbb{1}[N_{\underline{\delta}} > r] P_E.
\end{aligned}
$$

fix σ small enough such that

$$2\sigma\|a_{\underline{\delta}} G\|^2 \leq \epsilon/5. \quad\quad (4.62)$$

Fix now r large enough such that $\|P_E N_{\underline{\delta}} \mathbb{1}[N_{\underline{\delta}} > r] P_E\| \leq \epsilon/(5(2 + \sigma^{-1}))$. We then have

$$2\mathrm{Re}\{P_E N_{\underline{\delta}} \mathbb{1}[N_{\underline{\delta}} > r]\overline{P}_E\} \geq -\sigma N_{\underline{\delta}} - \frac{\epsilon}{5}\mathbb{1}_{\mathcal{H}}.$$

Secondly we estimate for $\sigma' > 0$

$$
\begin{aligned}
2\mathrm{Re}\{K\overline{P}_E\} &= 2\mathrm{Re}\{K\mathbb{1}[|H - E| < \kappa]\overline{P}_E\} + 2\mathrm{Re}\{K\mathbb{1}[|H - E| > \kappa]\} \\
&\geq -(\epsilon/5 + \sigma')\mathbb{1} - \frac{\|K\|^2}{\sigma'}\mathbb{1}[|H - E| > \kappa].
\end{aligned}
$$

Here we chose $\kappa > 0$ small enough such that $\|K\mathbb{1}[|H - E| < \kappa]\overline{P}_E\| \leq \epsilon/5$. Picking $\sigma' = \epsilon/5$ yields

$$H' \geq -\frac{4\epsilon}{5}\mathbb{1}_{\mathcal{H}} - \sigma N_\delta - C\mathbb{1}[|H - E| > \kappa].$$

To get rid of the extra σN_δ we estimate using (2.22) (with $\sigma = 1/2$) and (4.62)

$$(1 + 2\sigma)H' \geq -\frac{4\epsilon}{5}\mathbb{1} - C\mathbb{1}[|H - E| > \kappa] + \sigma N_\delta - 2\sigma\phi(\imath a_{\underline{\delta}}G)$$
$$\geq -\epsilon\mathbb{1} - C\mathbb{1}[|H - E| > \kappa].$$

It remains to establish that one can choose $\underline{\delta}'$, κ and C such that the desired bound holds for all $E \in J$ and $\underline{\delta} \in \Delta(\underline{\delta}')$. We proceed by assuming, aiming for a contradiction, that given $\underline{\delta}'_n = (1/n, n)$, $\kappa_n = 1/n$ and $C_n = n$, there exists an energy E_n and $\underline{\delta}_n \in \Delta(\underline{\delta}'_n)$ such that the desired bound fails. By compactness of J, we can assume that E_n converges to some E_∞. Let $\underline{\delta}'_\infty$, κ_∞ and C_∞ be the constants just established to exist such that the bound holds true at E_∞ for any $\underline{\delta} \in \Delta(\underline{\delta}'_\infty)$. Picking n large enough such that

$$|E_\infty - E_n| < \kappa_\infty/2, \quad \kappa_n < \kappa_\infty/2, \quad C_n \geq C_\infty, \quad \underline{\delta}'_n \in \Delta(\underline{\delta}'_\infty)$$

sets us up with a contradiction since with $\underline{\delta} \in \Delta(\underline{\delta}'_n) \subset \Delta(\underline{\delta}'_\infty)$ we have

$$H' \geq -\epsilon\mathbb{1} - C_\infty\mathbb{1}[|H - E_\infty| \geq \kappa_\infty] \geq (1 - \epsilon)\mathbb{1} - C_n\mathbb{1}[|H - E_n| \geq \kappa_n].$$

$\square$

The following theorem, which appeared originally in [28, Thm. 7.12], states that a Mourre estimate holds at any fixed $E \in \mathbb{R}$. It holds also for confined small system, not necessarily finite dimensional, but the proof simplifies slightly here. Furthermore, since we do not need resolvents of H to control $\phi(a_{\underline{\delta}}G)$ but can do with resolvents of N, the version here in fact holds under slightly weaker IR assumptions on G.

Another special feature of finite dimensional small systems is that we can choose δ'_0 uniformly in energy. Indeed, we pick $\delta'_0 \in (0, 1]$ such that

$$d(\delta'_0) \geq \sup_{\underline{\delta} \in \Delta_0} \|a_{\underline{\delta}}G\|^2 + 1. \tag{4.63}$$

With this choice we have for all $\underline{\delta} \in \Delta((\delta'_0, 1))$ and $|\omega| \leq \delta_0$ that

$$m_{\underline{\delta}}(\omega) \geq m_{\underline{\delta}}(\delta_0) = d(\delta_0) \geq d(\delta'_0) \geq \sup_{\underline{\delta} \in \Delta_0} \|a_{\underline{\delta}}G\|^2 + 1. \tag{4.64}$$

Theorem 4.7. *Suppose* **(HG1)**. *Let $\epsilon > 0$ and $E \in \mathbb{R}$. There exist $\kappa > 0$, $C > 0$ and K a compact and self-adjoint operator, such that the form estimate on $\mathcal{D}(N)$*

$$H' \geq (1 - \epsilon)\mathbb{1}_\mathcal{H} - C\mathbb{1}[|H - E| \geq \kappa] - K$$

holds true for all $\underline{\delta} \in \Delta((\delta'_0, 1))$. Here δ'_0 is chosen such that (4.63) is satisfied.

Proof. Fix $E \in \mathbb{R}$ and $\epsilon < 0$. We only have something to prove if $E \geq \Sigma$. The proof goes by induction in energy and we assume the theorem holds true for all $E' \leq E - \delta'_0$ and $\epsilon' > 0$.

Write $P = |0\rangle\langle 0|$ and $P^\perp = \mathbb{1}_\mathcal{F} - P$ as projection operators on $\mathcal{F}$ or $\mathcal{H}$ (read as e.g. $\mathbb{1}_\mathcal{K} \otimes P$). In order to use geometric localization we need the extended Hilbert space $\mathcal{H}^{\mathrm{x}} = \mathcal{H} \otimes \mathcal{F}$ and the extended Hamiltonian $H^{\mathrm{x}} = H \otimes \mathbb{1}_\mathcal{F} + \mathbb{1}_\mathcal{H} \otimes H_{\mathrm{ph}}$. The extended commutator is

$$H^{\mathrm{x}'} = H' \otimes \mathbb{1}_\mathcal{F} + \mathbb{1}_\mathcal{H} \otimes N_{\underline{\delta}}$$

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as a self-adjoint operator on $\mathcal{D}(N^{\mathrm{x}})$, where $N^{\mathrm{x}} = N \otimes \mathbb{1}_{\mathcal{F}} + \mathbb{1}_{\mathcal{H}} \otimes N$.

Observe that if $S\colon \mathcal{D}(N^{1/2}) \to \mathcal{H}^{\mathrm{x}}$ is bounded then for any $\sigma > 0$ we have

$$
\begin{aligned}
\mathrm{Re}\big\{ \check{\Gamma}(j^R)^*(\mathbb{1}_{\mathcal{H}} \otimes P)S \big\} &= \mathrm{Re}\big\{ \Gamma(j_0^R)S_0 \big\} \\
&= \mathrm{Re}\big\{ \mathbb{1}[|H - E| > 1]\Gamma(j_0^R)S_0(N + \mathbb{1}_{\mathcal{H}})^{-1/2}(N + \mathbb{1}_{\mathcal{H}})^{1/2} \big\} - K_1 \\
&\geq -\sigma(N + \mathbb{1}_{\mathcal{H}}) - \sigma^{-1}\|S_0(N + \mathbb{1}_{\mathcal{H}})^{-1/2}\|^2\mathbb{1}[|H - E| > 1] - K_1.
\end{aligned}
$$

Here $S_0 = (\mathbb{1}_{\mathcal{K}} \otimes P)S\colon \mathcal{D}(N^{1/2}) \to \mathcal{H} \otimes \mathbb{C}$ is the first component of S and K_1 is compact.

The observation above implies that we can pick $R_0 > 0$ large enough such that for $R \geq R_0$ we have for $\sigma > 0$ as a form on $\mathcal{D}(N)$

$$
\begin{aligned}
H' &\geq \check{\Gamma}(j^R)^* H^{\mathrm{x}'}\check{\Gamma}(j^R) - \sigma(N + \mathbb{1}_{\mathcal{H}}) \\
&\geq \check{\Gamma}(j^R)^*(\mathbb{1}_{\mathcal{H}} \otimes P^\perp)H^{\mathrm{x}'}(\mathbb{1}_{\mathcal{H}} \otimes P^\perp)\check{\Gamma}(j^R) \\
&\qquad - \sigma(N + \mathbb{1}_{\mathcal{H}}) - C_2\mathbb{1}[|H - E| > 1] - K_2.
\end{aligned}
\tag{4.65}
$$

Here we employed the observation above with an $(R, \underline{\delta})$-dependent operator $S(R, \underline{\delta}) = -(\phi(\imath a_{\underline{\delta}}G) \otimes \mathbb{1}_{\mathcal{F}})\check{\Gamma}(j^R)$, which has $\|S_0(R, \underline{\delta})(N + \mathbb{1}_{\mathcal{H}})^{-1/2}\|$ bounded uniformly in $R \geq R_0$ and $\underline{\delta} \in \Delta_0$.

Fix σ such that

$$
\sigma \leq \frac{\epsilon}{5}, \quad 2\sigma\|a_{\underline{\delta}}G\|^2 \leq \frac{\epsilon}{5} \quad \text{and} \quad \frac{1 - \frac{4\epsilon}{5}}{1 + 2\sigma} \geq 1 - \epsilon.
\tag{4.66}
$$

We now employ the momentum partition of unity from the proof of Theorem 4.3, cf. (4.58) and (4.59). Let $F^{\delta_0} = (\mathbb{1}[|k| \geq \delta_0], \mathbb{1}[|k| < \delta_0])$ and recall that the map $\check{\Gamma}(F^{\delta_0})\colon \mathcal{F} \to \mathcal{F}_>^{\delta_0} \otimes \mathcal{F}_<^{\delta_0}$ is unitary. (For notational convenience below, we have switched the order of the interior and exterior regions.) Abbreviate $\widehat{\mathcal{H}}^{\mathrm{x}} = (\mathbb{1}_{\mathcal{H}} \otimes \check{\Gamma}(F^{\delta_0}))\mathcal{H}^{\mathrm{x}} = \mathcal{H} \otimes \mathcal{F}_>^{\delta_0} \otimes \mathcal{F}_<^{\delta_0}$. Compute the intertwining relations

$$
\check{\Gamma}(F^{\delta_0})P^\perp = \big(\mathbb{1}_{\mathcal{F}_>^{\delta_0}} \otimes P_<^\perp + P_>^\perp \otimes P_<\big)\check{\Gamma}(F^{\delta_0})
\tag{4.67}
$$

$$
\big(\mathbb{1}_{\mathcal{H}} \otimes \check{\Gamma}(F^{\delta_0})\big)H^{\mathrm{x}'} = \widehat{H}^{\mathrm{x}'}\big(\mathbb{1}_{\mathcal{H}} \otimes \check{\Gamma}(F^{\delta_0})\big)
\tag{4.68}
$$

$$
\big(\mathbb{1}_{\mathcal{H}} \otimes \check{\Gamma}(F^{\delta_0})\big)H^{\mathrm{x}} = \widehat{H}^{\mathrm{x}}\big(\mathbb{1}_{\mathcal{H}} \otimes \check{\Gamma}(F^{\delta_0})\big).
\tag{4.69}
$$

Here $P_{>/<}$ denote the orthogonal projections onto the vacuum sectors inside $\mathcal{F}_{>/<}^{\delta_0}$, and

$$
\begin{aligned}
\widehat{H}^{\mathrm{x}'} &= H_>^{\mathrm{x}'} \otimes \mathbb{1}_{\mathcal{F}_<^{\delta_0}} + \mathbb{1}_{\mathcal{H}} \otimes \mathbb{1}_{\mathcal{F}_>^{\delta_0}} \otimes N_{\underline{\delta}|\mathcal{F}_<^{\delta_0}}, \\
H_>^{\mathrm{x}'} &= H' \otimes \mathbb{1}_{\mathcal{F}_>^{\delta_0}} + \mathbb{1}_{\mathcal{H}} \otimes N_{\underline{\delta}|\mathcal{F}_>^{\delta_0}}, \\
\widehat{H}^{\mathrm{x}} &= H \otimes \mathbb{1}_{\mathcal{F}_>^{\delta_0}} \otimes \mathbb{1}_{\mathcal{F}_<^{\delta_0}} + \mathbb{1}_{\mathcal{H}} \otimes H_{\mathrm{ph}|\mathcal{F}_>^{\delta_0}} \otimes \mathbb{1}_{\mathcal{F}_<^{\delta_0}} + \mathbb{1}_{\mathcal{H}} \otimes \mathbb{1}_{\mathcal{F}_>^{\delta_0}} \otimes H_{\mathrm{ph}|\mathcal{F}_<^{\delta_0}}.
\end{aligned}
$$

Using that $H' \geq -\|a_{\underline{\delta}}G\|^2$, cf. (2.22), we estimate

$$
\big(\mathbb{1}_{\mathcal{H}} \otimes \mathbb{1}_{\mathcal{F}_>^{\delta_0}} \otimes P_<^\perp\big)\widehat{H}^{\mathrm{x}'} \geq \big(m(\delta_0) - \|a_{\underline{\delta}}G\|^2\big)\mathbb{1}_{\mathcal{H}} \otimes \mathbb{1}_{\mathcal{F}_>^{\delta_0}} \otimes P_<^\perp
\tag{4.70}
$$

and observe the identity

$$
\big(\mathbb{1}_{\mathcal{H}} \otimes P_>^\perp \otimes P_<\big)\widehat{H}^{\mathrm{x}'} = \big((\mathbb{1}_{\mathcal{H}} \otimes P_>^\perp)H_>^{\mathrm{x}'}(\mathbb{1}_{\mathcal{H}} \otimes P_>^\perp)\big) \otimes P_<.
\tag{4.71}
$$

Using the intertwining relations (4.67) and (4.68), together with (4.70), (4.71) and the choice of δ_0', cf. (4.64), we get

$$
(\mathbb{1}_{\mathcal{H}} \otimes P^{\perp}) H^{x'} (\mathbb{1}_{\mathcal{H}} \otimes P^{\perp}) \geq \big(\mathbb{1}_{\mathcal{H}} \otimes \check{\Gamma}(F^{\delta_0})^* \big) \tag{4.72}
$$
$$
\times \Big\{ \mathbb{1}_{\mathcal{H}} \otimes \mathbb{1}_{\mathcal{F}_>^{\delta_0}} \otimes P_<^{\perp} + \big((\mathbb{1}_{\mathcal{H}} \otimes P_>^{\perp}) H_>^{x'} (\mathbb{1}_{\mathcal{H}} \otimes P_>^{\perp}) \big) \otimes P_< \Big\} \big(\mathbb{1}_{\mathcal{H}} \otimes \check{\Gamma}(F^{\delta_0}) \big).
$$

To deal with the term in the brackets we note that

$$
(\mathbb{1}_{\mathcal{H}} \otimes P_>^{\perp}) H_>^{x'} (\mathbb{1}_{\mathcal{H}} \otimes P_>^{\perp} \geq \mathbb{1}_{\mathcal{H}} \otimes P_>^{\perp} + H' \otimes P_>^{\perp} \tag{4.73}
$$

and estimate using Lemma 4.5, with ϵ replaced by $\epsilon/5$, and the induction assumption

$$
H' \otimes P_>^{\perp} \otimes P_< = \Big\{ \bigoplus_{\ell=1}^{\infty} \int_{(\mathbb{R}^3 \setminus B(\delta_0))^{\ell}}^{\oplus} H' \, \mathrm{d}k_1 \cdots \mathrm{d}k_{\ell} \Big\} \otimes P_<
$$
$$
\geq - \Big\{ \bigoplus_{\ell=1}^{\infty} \int_{(\mathbb{R}^3 \setminus B(\delta_0))^{\ell}}^{\oplus} \big(\epsilon \mathbb{1}_{\mathcal{H}} + C \mathbb{1}[|H + \textstyle\sum_{j=1}^{\ell} |k_j| - E| > \kappa] \big) \mathrm{d}k_1 \cdots \mathrm{d}k_{\ell} \Big\} \otimes P_<
$$
$$
= - \frac{\epsilon}{5} \mathbb{1}_{\mathcal{H}} \otimes P_>^{\perp} \otimes P_< - C \mathbb{1}[|\widehat{H}^x - E| > \kappa] \mathbb{1}_{\mathcal{H}} \otimes P_>^{\perp} \otimes P_<
$$
$$
\geq - \frac{\epsilon}{5} \mathbb{1}_{\widehat{\mathcal{H}}^x} - C \mathbb{1}[|\widehat{H}^x - E| > \kappa]. \tag{4.74}
$$

Here κ and C are coming from Lemma 4.5. Combining (4.72)–(4.74), cf. also (4.67) and (4.69), we find

$$
(\mathbb{1}_{\mathcal{H}} \otimes P^{\perp}) H^{x'} (\mathbb{1}_{\mathcal{H}} \otimes P^{\perp}) \geq \big(\mathbb{1}_{\mathcal{H}} \otimes \check{\Gamma}(F^{\delta_0})^* \big)
$$
$$
\times \Big\{ \mathbb{1}_{\mathcal{H}} \otimes \mathbb{1}_{\mathcal{F}_>^{\delta_0}} \otimes P_<^{\perp} + \mathbb{1}_{\mathcal{H}} \otimes P_>^{\perp} \otimes P_< - \frac{\epsilon}{5} \mathbb{1}_{\widehat{\mathcal{H}}^x} - C \mathbb{1}[|\widehat{H}^x - E| > \kappa] \Big\}
$$
$$
\times \big(\mathbb{1}_{\mathcal{H}} \otimes \check{\Gamma}(F^{\delta_0}) \big)
$$
$$
= \mathbb{1}_{\mathcal{H}} \otimes P^{\perp} - \frac{\epsilon}{5} \mathbb{1}_{\mathcal{H}^x} - C \mathbb{1}[|H^x - E| > \kappa].
$$

Pick a non-negative $f \in C_0^{\infty}(\mathbb{R})$ with $\mathrm{supp}(f) \subseteq [-\kappa, \kappa]$ and $f = 1$ on $[-\kappa/2, \kappa/2]$. We are now in a position to insert into (4.65) and estimate for some $R \geq R_0$, which we can now fix,

$$
H' \geq \check{\Gamma}(j^R)^* \big\{ \mathbb{1}_{\mathcal{H}} \otimes P^{\perp} - \epsilon \mathbb{1}_{\mathcal{H}^x} - C \mathbb{1}[|H^x - E| > \kappa] \big\} \check{\Gamma}(j^R)
$$
$$
- \sigma(N + \mathbb{1}_{\mathcal{H}}) - C_2 \mathbb{1}[|H - E| > 1] - K_2
$$
$$
\geq \check{\Gamma}(j^R)^* \big\{ \big(1 - \frac{\epsilon}{5}\big) \mathbb{1}_{\mathcal{H}^x} - C f(H^x - E) \big\} \check{\Gamma}(j^R)
$$
$$
- \sigma(N + \mathbb{1}_{\mathcal{H}}) - C_3 \mathbb{1}[|H - E| > 1] - K_3
$$
$$
\geq \big(1 - \frac{2\epsilon}{5}\big) \mathbb{1}_{\mathcal{H}} - \sigma(N + \mathbb{1}_{\mathcal{H}}) - C f(H - E) - C_3 \mathbb{1}[|H - E| > 1] - K_3
$$
$$
\geq \big(1 - \frac{2\epsilon}{5} - \sigma\big) \mathbb{1}_{\mathcal{H}} - \sigma N - C_4 \mathbb{1}[|H - E| > \kappa/2] - K_3.
$$

with $C_4 = C + C_3$.

The proof is now completed as in the proof of Lemma 4.5 by the bound

$$
(1 + 2\sigma) H' \geq H' + \sigma N - 2\sigma \| a_{\underline{\delta}} G \|^2 \geq \big(1 - \frac{4\epsilon}{5}\big) \mathbb{1}_{\mathcal{H}} - C_4 \mathbb{1}[|H - E| > \kappa/2] - K_3,
$$

where we used the choice of σ, cf. (4.66). This concludes the proof. $\qquad \square$

We arrive at the following structure result for the pure point spectrum of H.

Corollary 4.8. *Suppose* **(HG1)**. *The operator H has a finite number of eigenvalues, all of finite multiplicity.*

Proof. Assume towards a contradiction that there exists an enumerable sequence ψ_n of normalized eigenstates. Let μ_j denote the corresponding eigenvalues. Due to Theorems 3.6 and 4.3 we know that $\{\mu_j\}_{j=1}^{\infty}$ is a bounded sequence. Hence we can assume that it is convergent towards an energy E.

Now Theorems 3.6 and 4.7, applied with $\epsilon = 1/2$, yield the estimate

$$0 \geq \frac{1}{2} - \langle \psi_j, K\psi_j \rangle,$$

for $j \geq j_0$, where j_0 is such that $|\mu_j - E| < \kappa$ for $j \geq j_0$. Since K is compact and $\mathrm{w} - \lim \psi_j = 0$, we conclude that $\lim_{j \to \infty} \langle \psi_j, K\psi_j \rangle = 0$. This establishes the sought after contradiction. $\square$

In the following we denote by P the finite rank projection that projects onto the subspace consisting of eigenstates for H, and we write $\overline{P} = \mathbb{1} - P$.

Corollary 4.9. *Suppose* **(HG1)**. *Let $\epsilon > 0$. There exists $\underline{\delta}' \in \Delta_0$, $\kappa > 0$, $C > 0$, such that the following two estimates holds for all $\underline{\delta} \in \Delta(\underline{\delta}')$ and $E \in \mathbb{R}$*

$$H' \geq -\epsilon \mathbb{1} - C\mathbb{1}[|H - E| \geq \kappa], \tag{4.75}$$

$$H' \geq (1 - \epsilon)\mathbb{1} - C(\mathbb{1}[|H - E| \geq \kappa] + P), \tag{4.76}$$

in the sense of forms on $\mathcal{D}(N)$.

Proof. The estimate (4.75) is a direct consequence of Theorems 4.3 and 4.7 together with Lemma 4.5.

We proceed to the second bound (4.76). This bound is obviously true for $E > E_0 + 1$ (cf. Theorem 4.3) and for $E < \Sigma - 1$, so what remains is to prove the estimate uniformly in $E \in [\Sigma - 1, E_0 + 1] =: J$, which is a compact interval. We first argue that the estimate is correct for fixed $E \in J$. Let $\epsilon > 0$ and apply Theorem 4.7 with ϵ replaced by $\epsilon/3$. Write

$$K = PKP + \overline{P}K\overline{P} + 2\mathrm{Re}\{PK\overline{P}\} \geq -\|K\|P + \overline{P}K\overline{P} + 2\mathrm{Re}\{PK\overline{P}\}$$

and estimate

$$K\overline{P} = K\overline{P}\mathbb{1}[|H - E| \geq \kappa] + K\overline{P}\mathbb{1}[|H - E| < \kappa],$$

where one can choose κ small enough such that $\|K\overline{P}\mathbb{1}[|H - E| < \kappa]\| \leq \epsilon/3$. For the first term we estimate

$$2\langle \psi, K\overline{P}\mathbb{1}[|H - E| \geq \kappa]\psi \rangle \leq 2\|K\psi\|\|\mathbb{1}[|H - E| \geq \kappa]\psi\|$$

$$\leq \sigma\|K\|^2\|\psi\|^2 + \frac{1}{\sigma}\langle \psi, \mathbb{1}[|H - E| \geq \kappa]\psi \rangle.$$

Choosing $\sigma > 0$ small enough we get

$$K \geq -\tfrac{2}{3}\epsilon \mathbb{1} - \widetilde{C}\mathbb{1}[|H - E| \geq \kappa].$$

This completes the argument that for a fixed E one can find κ and C such that the commutator estimate (4.76) holds true.

Suppose the estimate (4.76) is not correct uniformly in E. That is, for any $\kappa > 0$ and $C > 0$ there exists $E \in J$ such that estimate fails to hold.

Put $\kappa_n = 1/n$ and $C_n = n$. This gives a sequence $E_n \in J$, for which the estimate (4.76) is false. We can assume due to compactness of J that E_n converges to an energy $E_\infty \in J$. Recalling that we have just verified that (4.76) holds for a fixed $E \in J$, we get a $\kappa_\infty > 0$ and $C_\infty > 0$ such that (4.76) holds true at E_∞. Pick n large enough such that $1/n < \kappa_\infty/2$, $C_n > C_\infty$ and $|E_\infty - E_n| < \kappa_\infty/2$. Then

$$H' \geq (1-\epsilon)\mathbb{1} - C_\infty(\mathbb{1}[|H-E| \geq \kappa_\infty] + P) \geq (1-\epsilon)\mathbb{1} - C_n(\mathbb{1}[|H-E_n| \geq \kappa_n] + P),$$

contradicting the choice of E_n. $\qquad\square$

4.4 Estimates at Positive Temperature

In this subsection we use the notation $\widetilde{N}_{\underline{\delta}}$ for $d\Gamma(m_{\underline{\delta}})$, which is the analogue of $N_{\underline{\delta}}$ from the previous subsection. We can then write $\widetilde{L}'_\beta = \mathbb{1}_{\mathcal{K}\otimes\mathcal{K}} \otimes \widetilde{N}_{\underline{\delta}} - \phi(\imath\tilde{a}_{\underline{\delta}}\widetilde{G}_\beta)$.

Theorem 4.10. *Suppose* **(LG1)**. *Let $e > 0$ be given. There exists $E_0 > 0$, $\delta'_\infty > 0$ and $C > 0$ such that the following form bound holds on $\mathcal{D}(N^{\mathrm{L}})$ for all $E \geq E_0$ and $\underline{\delta} \in \Delta((1, \delta'_\infty))$*

$$L'_\beta \geq e\mathbb{1} - C\mathbb{1}[|L_\beta| < E].$$

Proof. It suffices to prove the theorem, with L_β and N^{L} replaced by $\widetilde{L}_\beta$ and $\widetilde{N}$. The proof is divided into two steps. First we consider the uncoupled glued Liouvillean $\widetilde{L}_0$. The reader should not confuse the subscript 0 with infinite temperature (zero inverse temperature). We proceed as in the proof of Theorem 4.3.

Observe that

$$\widetilde{L}'_0 = \mathbb{1}_{\mathcal{K}\otimes\mathcal{K}} \otimes \widetilde{N}_{\underline{\delta}}.$$

For $R > 1$ we again perform a partition of unity in momentum space as follows. Let

$$j^R = \begin{pmatrix} \mathbb{1}[|\omega| < R] \\ \mathbb{1}[|\omega| \geq R] \end{pmatrix} : \tilde{\mathfrak{h}} \to \tilde{\mathfrak{h}}_< \oplus \tilde{\mathfrak{h}}_>$$

$$\tilde{\mathfrak{h}}_< := L^2((-R,R)) \otimes L^2(S^2), \quad \tilde{\mathfrak{h}}_> := L^2((-\infty,R] \cup [R,\infty)) \otimes L^2(S^2).$$

Put $\widetilde{\mathcal{F}}_< = \Gamma(\tilde{\mathfrak{h}}_<)$, $\widetilde{\mathcal{F}}_> = \Gamma(\tilde{\mathfrak{h}}_>)$ and

$$\widetilde{L}^{\mathsf{x}}_0 = L_K \otimes \mathbb{1}_{\widetilde{\mathcal{F}}_<\otimes\widetilde{\mathcal{F}}_>} + \mathbb{1}_{\mathcal{K}\otimes\mathcal{K}} \otimes d\Gamma(\omega)_{|\widetilde{\mathcal{F}}_<} \otimes \mathbb{1}_{\widetilde{\mathcal{F}}_>} + \mathbb{1}_{\mathcal{K}\otimes\mathcal{K}\otimes\widetilde{\mathcal{F}}_<} \otimes d\Gamma(\omega)_{|\widetilde{\mathcal{F}}_>}.$$

Abbreviate $\lambda_{\max} = \max\sigma(K)$ and $\lambda_{\min} = \min\sigma(K)$. We now estimate for $\widetilde{E} >$

$2\lambda_{\max} - \lambda_{\min}$:

$$\mathbb{1}_{\mathcal{K}\otimes\mathcal{K}} \otimes \widetilde{N}_{\underline{\delta}} \geq \mathbb{1}_{\mathcal{K}\otimes\mathcal{K}} \otimes \widetilde{N}_{\underline{\delta}}\mathbb{1}[|\widetilde{L}_0| + \widetilde{N} \geq \widetilde{E}]$$

$$= \check{\Gamma}(j^R)^* \left\{ \mathbb{1}_{\mathcal{K}\otimes\mathcal{K}} \otimes \widetilde{N}_{\underline{\delta}|\widetilde{\mathcal{F}}_<} \otimes \mathbb{1}_{\widetilde{\mathcal{F}}_>} + \mathbb{1}_{\mathcal{K}\otimes\mathcal{K}\otimes\widetilde{\mathcal{F}}_<} \otimes \widetilde{N}_{\underline{\delta}|\widetilde{\mathcal{F}}_>} \right\} \mathbb{1}[|\widetilde{L}_0^{\mathrm{x}}| + \widetilde{N}^{\mathrm{x}} \geq \widetilde{E}]\check{\Gamma}(j^R)$$

$$\geq \Gamma(\mathbb{1}[|\omega| < R])\widetilde{N}_{\underline{\delta}}\mathbb{1}[|\widetilde{L}_0| + \widetilde{N} \geq \widetilde{E}]$$

$$\quad + m_\delta(R)\check{\Gamma}(j^R)^*\mathbb{1}_{\mathcal{K}\otimes\mathcal{K}\otimes\widetilde{\mathcal{F}}_<} \otimes \overline{P}_\Omega\mathbb{1}[|\widetilde{L}_0^{\mathrm{x}}| + \widetilde{N}^{\mathrm{x}} \geq \widetilde{E}]\check{\Gamma}(j^R)$$

$$\geq \frac{\widetilde{E} - \lambda_{\max} + \lambda_{\min}}{R + 1}\Gamma(\mathbb{1}[|\omega| < R])\mathbb{1}[|\widetilde{L}_0| + \widetilde{N} \geq \widetilde{E}]$$

$$\quad + m_{\underline{\delta}}(R)\check{\Gamma}(j^R)^*\mathbb{1} \otimes \overline{P}_\Omega\mathbb{1}[|\widetilde{L}_0^{\mathrm{x}}| + \widetilde{N}^{\mathrm{x}} \geq \widetilde{E}]\check{\Gamma}(j^R)$$

$$\geq \min\left\{\frac{\widetilde{E} - \lambda_{\max} + \lambda_{\min}}{R + 1}, m_{\underline{\delta}}(R)\right\}\mathbb{1}[|\widetilde{L}_0| + \widetilde{N} \geq \widetilde{E}].$$

To conclude the proof we estimate

$$\mathbb{1}[|\widetilde{L}_0| + \widetilde{N} < \widetilde{E}] \leq (\widetilde{E} + 1)(|\widetilde{L}_0| + \widetilde{N} + 1)^{-1}$$

$$= (\widetilde{E} + 1)(|\widetilde{L}_\beta| + 1)^{-\frac{1}{2}}$$

$$\times \left\{ (|\widetilde{L}_\beta| + 1)^{\frac{1}{2}}(|\widetilde{L}_0| + \widetilde{N} + 1)^{-1}(|\widetilde{L}_\beta| + 1)^{\frac{1}{2}} \right\} (|\widetilde{L}_\beta| + 1)^{-\frac{1}{2}}$$

$$\leq C(\widetilde{E} + 1)\mathbb{1}[|\widetilde{L}_\beta| < E] + C\frac{\widetilde{E} + 1}{E + 1}.$$

Here we used Proposition 2.7 2 to establish that

$$C = \left\| (|\widetilde{L}_\beta| + 1)^{\frac{1}{2}}(|\widetilde{L}_0| + \widetilde{N} + 1)^{-1}(|\widetilde{L}_\beta| + 1)^{\frac{1}{2}} \right\| < \infty.$$

We can now pick R, δ', $\widetilde{E}$ and E in that order as in the proof of Theorem 4.3 to conclude the proof. $\qed$

It is now an immediate consequence of Theorem 3.7 that

Corollary 4.11. *Suppose* **(LG2)**. *The set of eigenvalues $\sigma_{\mathrm{pp}}(L_\beta)$ is bounded.*

From now on we assume at least **(HG1)** and fix $\underline{\delta}'$ such that Corollary 4.9 holds true. Recall that **(LG1)** implies **(HG1)**.

Proposition 4.12. *Suppose* **(HG1)**. *Let $\epsilon > 0$ be given. There exists $\kappa > 0$ and $C > 0$ such that for all $E \in \mathbb{R}$ and $\underline{\delta} \in \Delta(\underline{\delta}')$*

$$L'_\infty \geq (1 - \epsilon)\mathbb{1}_{\mathcal{H}^{\mathrm{L}}} - C\big(\mathbb{1}[|L_\infty - E| \geq \kappa] + P \otimes P^{\mathbf{c}}\big)$$

in the sense of forms on $\mathcal{D}(N^{\mathrm{L}})$.

Remark 4.13. Note that at zero temperature we do not need Nelson's commutator theorem to build L_∞, nor do we have any singularities from ρ_β to absorb. Hence we can work under an **(HG1)** condition instead of an **(LG1)** condition. $\Diamond$

Proof. The starting point is the identity

$$L'_\infty = H' \otimes \mathbb{1} + \mathbb{1} \otimes H^{\mathbf{c}'}.$$

Denote by $P \in \mathcal{B}(\mathcal{H})$ the projection onto the span of all eigenstates of the operator H. This is a finite range projection and hence compact. Put $P^{\mathbf{c}} = \mathbf{C}P\mathbf{C}$ to be the eigen projection onto the span of the eigenstates of $H^{\mathbf{c}}$. We write $\overline{P} = \mathbb{1} - P$ and $\overline{P^{\mathbf{c}}} = \mathbb{1} - P^{\mathbf{c}}$. We deal with $H' \otimes \mathbb{1}$ only since bounds on $\mathbb{1} \otimes H^{\mathbf{c}'}$ can be obtained by conjugation with $\mathcal{E}\mathbf{C}$, where $\mathcal{E}$ is the exchange map that sends $\psi \otimes \varphi$ to $\varphi \otimes \psi$. Here $\psi, \varphi \in \mathcal{H}$.

We write

$$H' = PH'P + 2\mathrm{Re}\{PH'\overline{P}\} + \overline{P}H'\overline{P}, \tag{4.77}$$

which makes sense as forms on $\mathcal{D}(N^{1/2})$ since P maps into $\mathcal{D}(N^{1/2})$ by Theorem 3.1. We estimate each term differently. For the first and last term we use (4.75) and (4.76) from Corollary 4.9 (applied with $\epsilon/9$ instead of ϵ) and find

$$PH'P \geq -\frac{\epsilon}{9}P - C\mathbb{1}[|H - \lambda - E| > \kappa] \geq -\frac{\epsilon}{9}\mathbb{1}_{\mathcal{H}} - C\mathbb{1}[|H - \lambda - E| > \kappa]$$
$$\overline{P}H'\overline{P} \geq \left(1 - \frac{\epsilon}{9}\right)\overline{P} - C\mathbb{1}[|H - \lambda - E| > \kappa]. \tag{4.78}$$

As for the cross term $PH'\overline{P}$ we proceed in a fashion similar to what was done in the proof of Lemma 4.5. Write for an $r > 0$

$$PH'\overline{P} = PN_{\underline{\delta}}\mathbb{1}[N_\delta > r]\overline{P} + K\overline{P},$$

with $K = PN_\delta\mathbb{1}[N_{\underline{\delta}} \leq r] - P\phi(\imath a_{\underline{\delta}}G)$ being compact. We can now fix first σ small enough and subsequently r large enough such that

$$2\mathrm{Re}\{PN_{\underline{\delta}}\mathbb{1}[N_\delta > r]\overline{P}\} \geq -\sigma N_{\underline{\delta}} - \frac{\epsilon}{9}\mathbb{1}_{\mathcal{H}}$$

and

$$2\sigma\|a_{\underline{\delta}}G\| \leq \frac{\epsilon}{9}, \qquad \frac{1 - \frac{8\epsilon}{9}}{1 + 2\sigma} > 1 - \epsilon. \tag{4.79}$$

To deal with the term $K\overline{P}$ we note that we can choose κ small enough such that $2\|K\overline{P}\mathbb{1}[|H - \lambda| < \kappa]\| \leq \epsilon/18$ uniformly in λ. Indeed, there exists Λ such that $2\|K\overline{P}\mathbb{1}[|H| > \Lambda]\| \leq \epsilon/18$ and hence by a covering argument there exists $\kappa > 0$ such that $2\|K\overline{P}\mathbb{1}[|H - \lambda| < \kappa]\| \leq \epsilon/18$ uniformly in $\lambda \in \mathbb{R}$. We thus get for all $\lambda, E \in \mathbb{R}$.

$$2\mathrm{Re}\{K\overline{P}\} = 2\mathrm{Re}\{K\overline{P}\mathbb{1}[|H - \lambda - E| > \kappa]\} + 2\mathrm{Re}\{K\overline{P}\mathbb{1}[|H - \lambda - E| < \kappa]\}$$
$$\geq -\frac{\epsilon}{18}\mathbb{1}_{\mathcal{H}} - \frac{C}{\epsilon}\mathbb{1}[|H - \lambda - E| > \kappa] - \frac{\epsilon}{18}\mathbb{1}_{\mathcal{H}}.$$

Inserting this together with (4.78) into (4.77) we arrive at the bound

$$H' \geq \left(1 - \frac{\epsilon}{9}\right)\overline{P} - \frac{3\epsilon}{9}\mathbb{1}_{\mathcal{H}} - \sigma N_{\underline{\delta}} - C\mathbb{1}[|H - \lambda - E| > \kappa]. \tag{4.80}$$

From the spectral theorem in multiplication operator form, we get a measure space $(\mathcal{M}, \Sigma, \mu)$, a measurable real function f on $\mathcal{M}$ and a unitary map $U\colon \mathcal{H} \to L^2(\mathcal{M})$ such that $UHU^* = M_f$, multiplication by f. Put $U^{\mathbf{c}} = U\mathbf{C}$ such that $U^{\mathbf{c}}H^{\mathbf{c}}U^{\mathbf{c}*} = M_f$ as well. Here $U^{\mathbf{c}*} = \mathbf{C}U^*$. The combined map $U^{\mathrm{L}} = U \otimes U^{\mathbf{c}}\colon \mathcal{H}^{\mathrm{L}} \to L^2(\mathcal{M} \times \mathcal{M})$ (with product σ-algebra and measure) now sets up the correspondence $U^{\mathrm{L}}L_\infty U^{\mathrm{L}*} = M_{f_1 - f_2}$, where $f_j(q_1, q_2) = f(q_j)$.

Then under the identification $L^2(\mathcal{M} \times \mathcal{M}) = L^2(\mathcal{M}; L^2(\mathcal{M}))$ we get

$$\mathbb{1}[|M_{f_1 - f_2} - E| > \kappa] = \int_{\mathcal{M}}^{\oplus} \mathbb{1}[|M_f - f(q) - E| > \kappa]\,\mathrm{d}\mu(q).$$

Hence we conclude from (4.80) the estimate

$$U^{\mathrm{L}}H' \otimes \mathbb{1}_{\mathcal{H}}U^{\mathrm{L}*} = \int_{\mathcal{M}}^{\oplus} UH'U^*\mathrm{d}\mu(q)$$

$$\geq \int_{\mathcal{M}}^{\oplus} \big(1-\frac{\epsilon}{9}\big)U\overline{P}U^* - \frac{3\epsilon}{9}\mathbb{1}_{L^2(\mathcal{M})} - \sigma UN_{\underline{\delta}}U^* - C\mathbb{1}[|M_f - f(q) - E| > \kappa]\,\mathrm{d}\mu(q)$$

$$= U^{\mathrm{L}}\Big(\big(1-\frac{\epsilon}{9}\big)\overline{P}\otimes\mathbb{1}_{\mathcal{H}} - \frac{3\epsilon}{9}\mathbb{1}_{\mathcal{H}^{\mathrm{L}}} - \sigma N_{\underline{\delta}}\otimes\mathbb{1}_{\mathcal{H}} - C\mathbb{1}[|L_\infty - E| > \kappa]\Big)U^{\mathrm{L}*}$$

in the sense of forms on $U^{\mathrm{L}}\mathcal{D}(N^{\mathrm{L}})$. Adding to the above a similar bound for $\mathbb{1}_{\mathcal{H}} \otimes H^{\mathbf{c}'}$ yields

$$L'_\infty \geq \big(1-\frac{\epsilon}{9}\big)\big[\overline{P}\otimes\mathbb{1}_{\mathcal{H}} + \mathbb{1}_{\mathcal{H}}\otimes\overline{P}^{\mathbf{c}}\big] - \frac{6\epsilon}{9}\mathbb{1}_{\mathcal{H}^{\mathrm{L}}}$$

$$- \sigma N^{\mathrm{L}} - C\big(\mathbb{1}[|L_\infty - E| > \kappa] + P\otimes P^{\mathbf{c}}\big)$$

$$\geq \big(1-\frac{7\epsilon}{9}\big)\mathbb{1}_{\mathcal{H}^{\mathrm{L}}} - \sigma N^{\mathrm{L}}_{\underline{\delta}} - (C+2)\big(\mathbb{1}[|L_\infty - E| > \kappa] + P\otimes P^{\mathbf{c}}\big).$$

Here we used that

$$\overline{P}\otimes\mathbb{1}_{\mathcal{H}} + \mathbb{1}_{\mathcal{H}}\otimes\overline{P}^{\mathbf{c}} = 2\mathbb{1}_{\mathcal{H}^{\mathrm{L}}} - P\otimes\overline{P}^{\mathbf{c}} - \overline{P}\otimes P^{\mathbf{c}} - 2P\otimes P^{\mathbf{c}} \geq \mathbb{1}_{\mathcal{H}^{\mathrm{L}}} - 2P\otimes P^{\mathbf{c}}.$$

We now complete the proof, cf. (4.79), by estimating

$$(1+2\sigma)L'_\infty \geq \big(1-\frac{8\epsilon}{9}\big)\mathbb{1}_{\mathcal{H}^{\mathrm{L}}} - C\big(\mathbb{1}[|L_\infty - E| > \kappa] + P\otimes P^{\mathbf{c}}\big).$$

as at the end of the proofs of Lemma 4.5 and Theorem 4.7. $\qquad\square$

In order to perturb around zero temperature, we first need to control the difference $\widetilde{G}_\beta - \widetilde{G}_\infty$.

Lemma 4.14. *Suppose* **(LGn)**, *for some $n \geq 0$. For any $\beta_0 > 0$ there exists $C > 0$ such that for all $\beta \geq \beta_0$ we have*

$$\big\|\widetilde{G}_\beta - \widetilde{G}_\infty\big\| \leq C\beta^{-\frac{1}{2}}. \tag{4.81}$$

If $n \geq 1$, we have furthermore that for all $\underline{\delta} \in \Delta_0$

$$\big\|\tilde{a}_{\underline{\delta}}\big(\widetilde{G}_\beta - \widetilde{G}_\infty\big)\big\| \leq C\beta^{-\frac{1}{2}}. \tag{4.82}$$

Proof. We begin with (4.81). For simplicity we only consider the term $(\sqrt{1+\widetilde{\rho}_\beta} - 1)\widetilde{G}_\infty$ in the expression for $\widetilde{G}_\beta$, cf. (2.40). The other term $\sqrt{\overline{\widetilde{\rho}_\beta}}\widetilde{G}^*_{\infty,\mathcal{R}}$ can be dealt with in a similar fashion.

Suppose an **(LGn)** condition, with $n \geq 0$. We split into the infrared and ultraviolet regimes and estimate first for $|\omega| \leq 1$:

$$\big(\sqrt{1+\widetilde{\rho}_\beta(\omega)} - 1\big)^2\big|\widetilde{G}_\infty(\omega,\Theta)\big|^2 \leq C\big(\sqrt{1+\widetilde{\rho}_\beta(\omega)} - 1\big)^2|\omega|^{2n+2\mu}.$$

Hence we can bound the L^2-norm squared of the contribution by a multiple of

$$\int_0^1 \big(\sqrt{1+\widetilde{\rho}_\beta(\omega)} - 1\big)^2\omega^{2n+2\mu}\,\mathrm{d}\omega \leq \beta^{-1}\int_0^1 (1+\omega)\omega^{2n-1+2\mu}\,\mathrm{d}\omega,$$

where we simply discarded the -1 term coming from $\widetilde{G}_\infty$. The integral is finite for all $n \geq 0$. In fact, the effect of subtracting $\widetilde{G}_\infty$ sits in the ultraviolet part where $|\omega| \geq 1$. Here we estimate the L^2-norm squared by

$$\int_1^\infty \big(\sqrt{1+\widetilde{\rho}_\beta(\omega)} - 1\big)^2 \omega^{-1-2\mu}\,\mathrm{d}\omega \leq \frac{\big(\sqrt{1+\widetilde{\rho}_\beta(1)} - 1\big)^2}{2\mu}.$$

Since $\sqrt{1+\widetilde{\rho}_\beta(1)} - 1 = \sqrt{1/(1-\mathrm{e}^{-\beta})} - 1 \sim \mathrm{e}^{-\beta/2}$ in the limit of large β, we get for a fixed $\beta_0 > 0$ a constant $C = C(\beta_0)$ such that for all $\beta > \beta_0$ we have (4.81) satisfied.

To establish (4.82) we observe that

$$\tilde{a}_{\underline{\delta}}\widetilde{G}_\beta = \big(\sqrt{1+\widetilde{\rho}_\beta} - 1\big)\tilde{a}_{\underline{\delta}}\widetilde{G}_\infty + m_{\underline{\delta}}\widetilde{G}_\infty \frac{\partial \sqrt{1+\widetilde{\rho}_\beta}}{\partial \omega}.$$

The first contribution can be estimate exactly as above, using that $n \geq 1$, and yields an $O(\beta^{-1/2})$ term. For the second term we compute

$$\frac{\partial \sqrt{1+\widetilde{\rho}_\beta}}{\partial \omega} = -\frac{\beta}{2}\widetilde{\rho}_\beta\sqrt{1+\widetilde{\rho}_\beta}.$$

In the infrared regime this can be dealt with easily since $\beta\widetilde{\rho}_\beta \leq 1/|\omega|$ and the extra inverse power of ω can be absorbed into $\widetilde{G}_\infty$. Recall that we assume $n \geq 1$. For the ultraviolet regime we get exponential decay in β from $\widetilde{\rho}_\beta(1)$ and we are done. $\square$

We remark that a similar bound holds for $\tilde{a}_{\underline{\delta}}^2(\widetilde{G}_\beta - \widetilde{G}_\infty)$ under an **(LG2)** condition but we do not need this.

Theorem 4.15. *Suppose* **(LG1)***. Let $\epsilon > 0$ be given. There exists $\beta_0 > 0$, $\kappa > 0$ and $C > 0$ such that for all $E \in \mathbb{R}$, $\underline{\delta} \in \Delta(\underline{\delta}')$, and $\beta \geq \beta_0$*

$$L'_\beta \geq (1-\epsilon)\mathbb{1} - C\big(\mathbb{1}[|L_\beta - E| \geq \kappa] + P \otimes P^{\mathbf{c}}\big)$$

in the sense of forms on $\mathcal{D}(N^{\mathrm{L}})$.

Proof. From (4.56), Proposition 4.12, applied with $\epsilon/4$ instead of ϵ, and Lemma 4.14 we get as a form bound on $\mathcal{D}(\widetilde{N})$

$$\widetilde{L}'_\beta = \widetilde{L}'_\infty - \phi\big(\imath\tilde{a}_{\underline{\delta}}(\widetilde{G}_\beta - \widetilde{G}_\infty)\big)$$
$$\geq \big(1 - \frac{\epsilon}{4} - \frac{C_1}{\sigma\beta}\big)\mathbb{1}_{\widetilde{\mathcal{H}}^{\mathrm{L}}} - \sigma\widetilde{N} - C\big(\mathbb{1}[|\widetilde{L}_\infty - E| > \kappa] + P_\infty\big),$$

valid for all $\sigma > 0$. Here $P_\infty = \mathcal{U}(P \otimes P^{\mathbf{c}})\mathcal{U}^*$ is the projection onto the eigenstates of $\widetilde{L}_\infty$.

Pick a non-negative $f \in C_0^\infty(\mathbb{R})$ with $\mathrm{supp}(f) \subseteq [-\kappa, \kappa]$ and $f = 1$ on $[-\kappa/2, \kappa/2]$. Let $\tilde{f}$ be an almost analytic extension of f. Write

$$f(\widetilde{L}_\infty - E) - f(\widetilde{L}_\beta - E) = \frac{1}{\pi}\int_{\mathbb{C}} \bar{\partial}\tilde{f}(\eta)\big((\widetilde{L}_\infty - \eta)^{-1} - (\widetilde{L}_\beta - \eta)^{-1}\big)\,\mathrm{d}\eta.$$

Since $\widetilde{N}$ is of class $C^1(\widetilde{L}_\beta)$ with $[\widetilde{N}, \widetilde{L}_\beta]^\circ$ being $\sqrt{\widetilde{N}}$-bounded, cf. Corollary 2.9, we conclude from Proposition A.13 2 (and interpolation) that $(\widetilde{L}_\beta - \eta)^{-1}$ preserves $\mathcal{D}(\sqrt{\widetilde{N}})$ and that there exists n and C such that

$$\big\|(\widetilde{N} + 1)^{\frac{1}{2}}(\widetilde{L}_\beta - \eta)^{-1}(\widetilde{N} + 1)^{-\frac{1}{2}}\big\| \leq C\big(1 + |\mathrm{Im}\eta|^{-n}\big).$$

It follows that

$$\big\|\big(f(\widetilde{L}_\infty - E) - f(\widetilde{L}_\beta - E)\big)(\widetilde{N} + 1)^{-\frac{1}{2}}\big\|$$

$$\leq \frac{1}{\pi} \int_{\mathbb{C}} |\bar{\partial}\tilde{f}(\eta)| \, |\mathrm{Im}\eta|^{-1} \big\|\phi\big(\widetilde{G}_\beta - \widetilde{G}_\infty\big)(\widetilde{L}_\beta - \eta)^{-1}(\widetilde{N} + 1)^{-\frac{1}{2}}\big\| \, \mathrm{d}\eta$$

$$\leq C \big\|\phi\big(\widetilde{G}_\beta - \widetilde{G}_\infty\big)(\widetilde{N} + 1)^{-\frac{1}{2}}\big\|.$$

Appealing to Lemma 4.14 we thus get

$$\mathbb{1}[|\widetilde{L}_\infty - E| > \kappa] \leq \mathbb{1}_{\widetilde{\mathcal{H}}^{\mathrm{L}}} - f(\widetilde{L}_\infty - E)$$

$$\leq \mathbb{1}_{\widetilde{\mathcal{H}}^{\mathrm{L}}} - f(\widetilde{L}_\beta - E) + \sigma\widetilde{N} + \frac{C_2}{\sigma\beta}$$

$$\leq \mathbb{1}[|\widetilde{L}_\beta - E| > \kappa/2] + \sigma\widetilde{N} + \frac{C_2}{\sigma\beta}.$$

Choose first $\sigma > 0$ small enough such that

$$3\sigma \sup_{\beta \geq 1, \underline{\delta} \in \Delta_0} \big\|\tilde{a}_{\underline{\delta}}\widetilde{G}_\beta\big\| < \frac{\epsilon}{4} \quad \text{and} \quad \frac{1 - 3\epsilon/4}{1 + 3\sigma} > 1 - \epsilon, \tag{4.83}$$

and subsequently $\beta_0 \geq 1$ large enough such that

$$\frac{C_1}{\sigma\beta_0} + \frac{C_2}{\sigma\beta_0} < \frac{\epsilon}{4}.$$

With these choices we arrive at the bound

$$\widetilde{L}'_\beta \geq \big(1 - \frac{\epsilon}{2}\big)\mathbb{1}_{\widetilde{\mathcal{H}}^{\mathrm{L}}} - 2\sigma\widetilde{N} - C\big(\mathbb{1}[|\widetilde{L}_\beta - E| \geq \kappa/2] + P_\infty\big).$$

We conclude the proof by the usual argument, i.e. bounding $(1 + 3\sigma)\widetilde{L}'_\beta$ from below, cf. (4.83) and the previous proof. $\square$

We conclude, repeating the proof of Corollary 4.8,

Corollary 4.16. *Suppose* **(LG2)***. There exists $\beta_0 > 0$ such that for all $\beta \geq \beta_0$, the Liouvillean L_β has finitely many eigenvalues, all of finite multiplicity.*

We remark that in a (β, G)-regime where a positive commutator estimate holds we can under the **(LG2)** condition conclude that eigenstates ψ of the standard Liouvillean L_β satisfy that $\psi \in \mathcal{D}(N^{\mathrm{L}})$. This is a consequence of [19] and improves the basic number bound Theorem 3.5, without imposing further conditions on G.

4.5 Open Problems IV

The by far most central open question relevant for this section is whether or not one can establish a positive commutator estimate for the standard Liouvillean for arbitrary inverse temperature β and coupling G. We have an unsubstantiated inkling that it should be possible to use $\widetilde{A}_{\underline{\delta}}$.

Problem 4.1. Establish, for arbitrary β and G, a positive commutator estimate for the Jakšić-Pillet glued standard Liouvillean $\widetilde{L}_\beta$, possibly making use of the conjugate operator $\widetilde{A}_{\underline{\delta}}$. It would be natural to work under the assumption **(LG2)**, and indeed we expect that this assumption should suffice.

We remark that we have not in this section made use of the modular conjugation J, cf. (2.34), which takes L_β to $-L_\beta$. This may be an extra ingredient to make use of.

When establishing positive commutator estimates in this section, either at weak coupling, high energy or low temperature, we did not attempt to determine a joint (β, G, E)-regime in which one can get a positive commutator. Without a positive answer to Problem 4.1, investigating the interplay between the different approaches above would be natural.

Problem 4.2. Determine a joint (β, G, E)-regime where one can derive a positive commutator estimate.

While one can establish positive commutator estimates for the Hamiltonian also for infinite dimensional small systems, cf. [28], the situation is fundamentally different for standard Liouvilleans. To see this consider as the small system a one-dimensional harmonic oscillator. Here the uncoupled Liouvillean L_0 will have point spectrum (a multiple of) $\mathbb{Z}$, with each eigenvalue having infinite multiplicity. Hence, one should not expect a positive commutator estimate with compact error terms, barring some mechanism to lift the infinite degeneracy by other means. However, in the dipole approximation this model is explicitly solvable [5, 6] and Könenberg in his thesis managed to handle perturbations of the Harmonic oscillator potential [43]. Note that one can construct a small system where the Hamiltonian K has compact resolvent and L_0 has point spectrum which is dense in $\mathbb{R}$! See also [24, 25] where an atomic small system is considered, and positive commutator methods are applied in the weak coupling regime.

Problem 4.3. What can be said about the general structure of the point spectrum of L_β, without the assumption of small coupling or a finite dimensional small system. Are positive commutator estimates useful at all?

We emphasize that all the proofs from Subsect. 4.4 make essential use of $\mathcal{K}$ being finite dimensional.

5 Absence of Singular Continuous Spectrum

The aim of this section is to establish the following two theorems about absence of singular continuous spectrum of Pauli-Fierz Hamiltonians at zero and positive temperature.

In this section we impose an **(HG2)** assumption at zero temperature and an **(LG2)** assumption at positive temperature. We will need an extra ultraviolet assumption, which we found inconvenient to include in **(HG2)** and **(LG2)**. It reads

$$\textbf{(SC)} \qquad (|k| + 1)\partial_j G \in L^2(\mathbb{R}^3; \mathrm{M}_\nu(\mathbb{C})),$$

for $j = 1, 2, 3$.

Theorem 5.1. *Suppose* **(HG2)** *and* **(SC)**. *Then* $\sigma_{\mathrm{sc}}(H) = \emptyset$.

The above theorem is due to [28], but our reproduction here establishes the theorem under slightly weaker assumptions, a consequence of our choice of small system as finite dimensional.

The following positive temperature analogue however is new and improves on a small coupling result going back to [15, 47].

Theorem 5.2. *Suppose* **(LG2)** *and* **(SC)**. *Then the following holds*

1. *There exists $\beta_0 > 0$ such that $\sigma_{\mathrm{sc}}(L_\beta) = \emptyset$, for all $\beta > \beta_0$.*

2. *For any β there exists $\Lambda > 0$ such that $\sigma_{\mathrm{sc}}(L_\beta) \subset [-\Lambda, \Lambda]$.*

Together with Theorem 3.11, this reduces return to equilibrium at low temperature and arbitrary coupling strength, to establishing that the zero eigenvalue of L_β is simple.

As usual, the road we take to establish Theorems 5.1 and 5.2. passes through a Limiting Absorption Principle (LAP). While we can employ the LAP from [27] to deal with the zero-temperature case, there is no LAP available in the literature which can deal with the positive temperature Liouvillean, outside the weak coupling regime where [15, 47] apply. In the following two subsections we establish a new LAP, which applies at both zero and positive temperature.

5.1 A priori Resolvent Estimates

In this subsection we work under the following assumptions on two self-adjoint operators T and T' acting on a Hilbert space $\mathcal{H}$.

(M1) T' is of class $C^1_{\mathrm{Mo}}(T)$, with $[T, T']^0 \in \mathcal{B}(\mathcal{D}(|T'|^{1/2}); \mathcal{H})$.

(M2) There exists $e, C_M > 0$ and $J \subset \mathbb{R}$ an open interval such that $T' \geq e\mathbb{1} - C_M \mathbb{1}[T \in J]$.

The condition **(M2)** in particular ensures that T' is semibounded. We fix in the following a real number η such that $T' + \eta \geq 1$.

Let J' be a compact subinterval of J. Pick $\kappa > 0$ such that $J'_\kappa = J' + [-\kappa, \kappa] \subset J$. From Urysohn's lemma we get an $f \in C_0^\infty(\mathbb{R}; [0, 1])$, with $\mathrm{supp}(f) \subset J$ and $f(t) = 1$ for $t \in J'_\kappa$. Then, by **(M2)**,

$$M := T' + C_M f^\perp(T) \geq e\mathbb{1}, \tag{5.84}$$

where $f^\perp = 1 - f$. Note that $\mathcal{D}(M) = \mathcal{D}(T')$. Put $\mathcal{M} = \mathcal{D}(M^{1/2}) = \mathcal{D}((T' + \eta)^{1/2})$ and equip $\mathcal{M}$ with the norm $\|u\|_{\mathcal{M}} = \sqrt{\langle u, Mu \rangle}$, with respect to which it is complete. Note that with this notation (M1) implies $[T, T']^0 \in \mathcal{B}(\mathcal{M}; \mathcal{H})$ and hence by duality we have $[T, T']^0 \in \mathcal{B}(\mathcal{H}; \mathcal{M}^*)$ as well.

Remarks 5.3.   1. Put $\mathcal{D} = \mathcal{D}(T) \cap \mathcal{D}(T')$. For $\epsilon \in \mathbb{R}$ the operator $T_\epsilon = T - \imath\epsilon T'$ is a priori defined as an operator on $\mathcal{D}$. By Proposition A.10, T_ϵ is in fact closed for $\epsilon \neq 0$ and $T_\epsilon^* = T_{-\epsilon}$. A fact also exploited in the proof of Proposition 2.7.

   2. An application of Proposition A.13 3, with $A = T$, $S = T' + \eta\mathbb{1}$, and $\rho = 1/2$, shows that for all $f \in C_0^\infty(\mathbb{R})$ we have $f(T) \colon \mathcal{D}(T') \to \mathcal{D}(T')$ continuously. By interpolation we also find that $f(T) \in \mathcal{B}(\mathcal{M})$.

This section is devoted to the study of the resolvent set of T_ϵ, and to establish bounds on the corresponding resolvents. It is precisely operators on the form T_ϵ that enter into Mourre's differential inequality technique to establish Limiting Absorption Principles. Our particular construction appeared first in [27], and is closely related to constructions from [51, 62].

Lemma 5.4. *There exists $C_1, C_2 > 0$ such that for any $\epsilon \in \mathbb{R}$ and $z \in \mathbb{C}$, with $\mathrm{Re}(z) \in J'$, we have*

$$\forall u \in \mathcal{D} : \quad \|\langle T \rangle f^\perp(T) u\| \leq C_1 \|(T_\epsilon - z)u\| + |\epsilon|^{\frac{1}{2}} C_2 \|u\|_{\mathcal{M}}.$$

Proof. Compute first for $u \in \mathcal{D}$

$$\|(T_\epsilon - z)u\|^2 + \epsilon \langle u, [T, T']^0 u \rangle = \|(T - z)u\|^2 + \|(\epsilon T' + \mathrm{Im}(z))u\|^2.$$

Discarding the last term yields the bound

$$\|(T - \mathrm{Re}(z))u\|^2 \leq \|(T - z)u\|^2 \leq \|(T_\epsilon - z)u\|^2 + |\epsilon|C\|u\|_{\mathcal{M}}^2.$$

Let $v \in \mathcal{D}$. Inserting $u = f^\perp(T)v \in \mathcal{D}$, cf. Remark 5.3 2, gives the bound

$$C_\kappa^2 \|\langle T \rangle f^\perp(T)v\|^2 \leq \|(T_\epsilon - z)f^\perp(T)v\|^2 + |\epsilon|C\|v\|_{\mathcal{M}}^2,$$

where $C_\kappa = \inf_{\lambda \in \mathbb{R} \setminus J'_\kappa, \mu \in J'} |\lambda - \mu|/\langle \lambda \rangle > 0$. Here we used that $M^{1/2} f(T) M^{-1/2}$ is bounded, again appealing to Remark 5.3 2. Consider the term

$$(T_\epsilon - z)f^\perp(T)v = f^\perp(T)(T_\epsilon - z)v - \imath\epsilon[T', f(T)]v.$$

The last term is bounded by $|\epsilon|\|v\|_{\mathcal{M}}$, which follows from (M1) by writing $f(T)$ as an integral over resolvents using an almost analytic extension of f. Here it is important that $[T, T']^0 \in \mathcal{B}(\mathcal{M}; \mathcal{H})$. This completes the proof by subadditivity of the square root. $\qquad\square$

Before continuing we observe that (M2) implies the following elementary bound, which holds for all ϵ and z with $\epsilon\mathrm{Im}(z) \geq 0$:

$$\forall u \in \mathcal{D} : \quad |\epsilon|\|u\|_{\mathcal{M}}^2 + |\mathrm{Im}(z)|\|u\|^2 \leq |\mathrm{Im}\langle u, (T_\epsilon - z)u \rangle| + |\epsilon|C_M\|f^\perp(T)u\|^2. \tag{5.85}$$

Lemma 5.5. *There exists $\epsilon_0 > 0$ and $C > 0$ such that for $\epsilon \in \mathbb{R}$ and $z \in \mathbb{C}$, with $0 < |\epsilon| \leq \epsilon_0$, $\mathrm{Re}(z) \in J'$ and $\epsilon\mathrm{Im}(z) > 0$, we have $z \in \rho(T_\epsilon)$ and the corresponding resolvent $R_\epsilon(z) = (T_\epsilon - z)^{-1}$ satisfies*

$$\|R_\epsilon(z)\| \leq C|e\epsilon + \mathrm{Im}(z)|^{-1}.$$

Proof. It suffices to consider the case $\epsilon > 0$, and hence we have also $\text{Im}(z) > 0$. From (5.85) we get

$$\epsilon\|u\|_{\mathcal{M}}^2 + \text{Im}(z)\|u\|^2 \leq |\langle u, (T_\epsilon - z)u\rangle| + \epsilon C_M\|f^\perp(T)u\|^2.$$

The last term we can estimate using Lemma 5.4 to obtain the bound

$$\|f^\perp(T)u\| \leq C_1\|(T_\epsilon - z)u\| + C_2\epsilon^{\frac{1}{2}}\|u\|_{\mathcal{M}}^2.$$

Hence we get

$$\epsilon\|u\|_{\mathcal{M}}^2 + \text{Im}(z)\|u\|^2 \leq (1 + \epsilon C_M C_1)\|u\|\|(T_\epsilon - z)u\| + \epsilon^{\frac{3}{2}}C_2 C_M\|u\|\|u\|_{\mathcal{M}}.$$

Estimate one factor of $\|u\|_{\mathcal{M}}$ from below by $\sqrt{e}\|u\|$, cf. (5.84), on the left-hand side and divide through by $\|u\|$. This yields the bound

$$\epsilon\sqrt{e}\|u\|_{\mathcal{M}} + \text{Im}(z)\|u\| \leq (1 + \epsilon C_M C_1)\|(T_\epsilon - z)u\| + \epsilon^{\frac{3}{2}}C_2 C_M\|u\|_{\mathcal{M}}.$$

Choosing now $\epsilon < \epsilon_0 := e(2C_2 C_M)^{-2}$ and making use of the lower bound $\|u\|_{\mathcal{M}} \geq \sqrt{e}\|u\|^2$ yet again concludes the proof. $\qquad\square$

From now on we keep the ϵ_0 from Lemma 5.5 fixed.

Lemma 5.6. *There exists $C > 0$ such that for all $\epsilon \in \mathbb{R}$, $z \in \mathbb{C}$ and $u \in \mathcal{H}$, with $0 < |\epsilon| \leq \epsilon_0$, $\text{Re}(z) \in J'$ and $\epsilon\text{Im}(z) > 0$, we have*

$$\|R_\epsilon(z)u\|_{\mathcal{M}} \leq C|\epsilon|^{-\frac{1}{2}}\|u\| \quad \text{and} \quad \|R_\epsilon(z)u\| \leq C|\epsilon|^{-\frac{1}{2}}\|u\|_{\mathcal{M}^*}.$$

Proof. It suffices to establish the bounds with $\epsilon > 0$ and hence $\text{Im}(z) > 0$. Furthermore, the first bound implies the second by duality.

Compute for $u \in \mathcal{H}$ with $\|u\| \leq 1$.

$$\|MR_\epsilon(z)u\| \leq \|T'R_\epsilon(z)u\| + C_M\|f^\perp(T)R_\epsilon(z)u\|. \tag{5.86}$$

As for the first term on the right-hand side of (5.86) we get using Lemma 5.4

$$\begin{aligned}
\|T'R_\epsilon(z)u\| &= \frac{1}{\epsilon}\|((T_\epsilon - z) - (T - z))R_\epsilon u\| \\
&\leq C\epsilon^{-1} + \epsilon^{-1}\|\langle T\rangle R_\epsilon(z)u\| \\
&\leq C'\epsilon^{-1} + \epsilon^{-1}\|\langle T\rangle f^\perp(T)R_\epsilon(z)u\|
\end{aligned}$$

Inserting back into (5.86) we arrive at

$$\begin{aligned}
\|MR_\epsilon(z)u\| &\leq C'\epsilon^{-1} + \epsilon^{-1}(1 + C_M)\|\langle T\rangle f^\perp(T)R_\epsilon(z)u\| \\
&\leq C''\epsilon^{-1} + C\|R_\epsilon(z)u\|_{\mathcal{M}} \\
&\leq C'''\epsilon^{-1} + \frac{1}{2}\|MR_\epsilon(z)u\|.
\end{aligned}$$

We conclude that

$$\|MR_\epsilon(z)u\| \leq 2C'''\epsilon^{-1},$$

and hence by interpolation

$$\|M^{\frac{1}{2}}R_\epsilon(z)u\| \leq \sqrt{2C'''}\epsilon^{-\frac{1}{2}}.$$

This concludes the proof $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 5.7. *There exists $C_1, C_2 > 0$ such that for all $\epsilon \in \mathbb{R}$ and $z \in \mathbb{C}$, with $0 < |\epsilon| \le \epsilon_0$, $\mathrm{Re}(z) \in J'$ and $\epsilon \mathrm{Im}(z) > 0$, we have*

1. *$\forall u \in \mathcal{H} : \quad |\epsilon|^{\frac{1}{2}} \|R_\epsilon(z)u\|_{\mathcal{M}} \le |\langle u, R_\epsilon(z)u\rangle|^{\frac{1}{2}} + C_1\|u\|_{\mathcal{M}^*}.$*

2. *$R_\epsilon(z)$ extends by continuity from $\mathcal{H}$ to an element of $\mathcal{B}(\mathcal{M}^*; \mathcal{M})$ and the resolvent satisfies the bound $\|R_\epsilon(z)\|_{\mathcal{B}(\mathcal{M}^*;\mathcal{M})} \le C_2|\epsilon|^{-1}.$*

Proof. Since for $u \in \mathcal{H}$ we have $R_\epsilon(z)u \in \mathcal{D}$ we get from (5.85) and Lemma 5.6 the estimate

$$\epsilon\|R_\epsilon(z)u\|_{\mathcal{M}}^2 \le |\mathrm{Im}\langle u, R_\epsilon(z)u\rangle| + \epsilon C_M\|f^\perp(T)R_\epsilon(z)u\|^2$$
$$\le |\mathrm{Im}\langle u, R_\epsilon(z)u\rangle| + C\|u\|_{\mathcal{M}^*}.$$

This implies 1.

To see 2 observe that

$$|\langle u, R_\epsilon(z)u\rangle| \le \|R_\epsilon(z)u\|_{\mathcal{M}}\|u\|_{\mathcal{M}^*} \le \frac{\epsilon}{2}\|R_\epsilon(z)u\|_{\mathcal{M}}^2 + \frac{1}{2\epsilon}\|u\|_{\mathcal{M}^*}^2.$$

From this estimate and 1, the statement 2 follows. $\qquad\square$

Proposition 5.8. *Let $z \in \mathbb{C}$, with $\mathrm{Re}(z) \in J'$ and $\mathrm{Im}(z) \ne 0$. We have the strong limit on $\mathcal{H}$*

$$\mathrm{s} - \lim_{\epsilon \to 0, \epsilon\mathrm{Im}(z) > 0} R_\epsilon(z) = (T - z)^{-1}.$$

Proof. Let $u \in \mathcal{D}(M) = \mathcal{D}(T')$ and compute the difference

$$(R_\epsilon(z) - (T - z)^{-1})u = \imath\epsilon(T - z)^{-1}T'R_\epsilon(z)u$$
$$= \imath\epsilon(T - z)^{-1}R_\epsilon(z)T'u + \imath\epsilon(T - z)^{-1}R_\epsilon(z)[T, T']^\circ R_\epsilon(z)u.$$

Since $[T, T']^\circ$ is $M^{1/2}$-bounded, cf. **(M1)**, it follows from Lemmata 5.5 and 5.6 that the right-hand side goes to zero, when $\epsilon \to 0$ while keeping $\epsilon\mathrm{Im}(z) > 0$.

This implies the result, since $\mathcal{D}(M)$ is dense in $\mathcal{H}$ and $R_\epsilon(z)$ is uniformly bounded in ϵ with $\epsilon\mathrm{Im}(z) > 0$. See Lemma 5.5. $\qquad\square$

5.2 Limiting Absorption Principle

Before we state and prove the main result of this section, the Limiting Absorption Principle, we need to impose conditions on how a conjugate operator A fits together with T and T'.

The operator A should be maximally symmetric, i.e. have at least one deficiency index equal to 0. To conform with the example of Pauli-Fierz Hamiltonians we assume $n_+ = \dim(\ker(A^* + \imath)) = 0$, such that $\{z \in \mathbb{C} \mid \mathrm{Im}(z) < 0\} \subset \rho(A)$. We write W_t for the semigroup of isometries generated by A, and W_t^* for the contraction semigroup generated by $-A^*$. A sub Hilbert space $\mathcal{G}$ of $\mathcal{H}$ is said to be b-preserved by a semigroup W_t if $W_t\mathcal{G} \subseteq \mathcal{G}$ and $\sup_{0 \le t \le 1} \|W_t u\|_{\mathcal{G}} < \infty$ for all $u \in \mathcal{G}$.

We add to **(M1)** and **(M2)** the following assumptions

(M3) $(T' + \eta)^{1/2}$ is of class $C^1_{\mathrm{Mo}}(A)$ and W_t^* b-preserves $\mathcal{M}$.

(M4) There exists $\sigma > 0$ such that for $\zeta \in \mathbb{C}$, with $\mathrm{Im}(\zeta) \ge \sigma$, we have

$$\forall u \in \mathcal{D} : \quad \langle u, \imath[(A + \zeta)^{-1}, T]u\rangle = \langle u, (A + \zeta)^{-1}T'(A + \zeta)^{-1}u\rangle.$$

Remarks 5.9. Let us make some observations pertaining to **(M3)** and **(M4)**:

1. The first part of assumption **(M3)**, together with Lemma A.8 3, ensures the existence of a $\sigma > 0$ such that $(A + \zeta)^{-1} \colon \mathcal{M} \to \mathcal{M}$, provided $\operatorname{Im}(\zeta) \geq \sigma$. The second part of assumption **(M3)** ensure that $(A^* + \bar{\zeta})^{-1} \colon \mathcal{M} \to \mathcal{M}$, provided $\operatorname{Im}(\zeta) \geq \sigma$, with a possibly larger σ. By duality $(A + \zeta)^{-1} \colon \mathcal{M}^* \to \mathcal{M}^*$ for $\operatorname{Im}(\zeta) > \sigma$. Hence both sides of the expression in **(M4)** makes sense for $\operatorname{Im}(\zeta)$ larger than this σ.

2. It is a consequence of **(M3)** that T' is of class $C^1(A)$ and $T'' := [T', A]^\circ \in \mathcal{B}(\mathcal{M}; \mathcal{M}^*)$. See Lemma 5.12 below for a proof. The converse is probably false although we do not have a counter example.

3. One can verify **(M3)**, by checking the stronger condition that T' is of class $C^1_{\mathrm{Mo}}(A)$ and $[T', A]^\circ \in \mathcal{B}(\mathcal{D}((T' + \eta)^\rho); \mathcal{H})$ for some $\rho \in [0, 1)$. See Proposition A.13 1.

4. In the following, σ refers to the constant from **(M4)**. The particular choice of the condition **(M4)** is inspired by [19].

We are now in a position to formulate the basic form of the Limiting Absorption Principle.

Theorem 5.10 (Limiting Absorption Principle). *Assume T, T' and A satisfies the assumptions* **(M1)**–**(M4)**. *Then there exists $C > 0$ such that for any $u \in \mathcal{D}(A)$ we have*

$$\sup_{\substack{z \in \mathbb{C} \\ \operatorname{Re}(z) \in J', \operatorname{Im}(z) \neq 0}} \left| \langle u, (T - z)^{-1} u \rangle \right| \leq C \left(\|u\|^2_{\mathcal{M}^*} + \|Au\|^2_{\mathcal{M}^*} \right).$$

Remark 5.11. We remark that, just as in [28], it suffices to assume that A generates a contraction semigroup. $\diamond$

Lemma 5.12. *Suppose* **(M3)**. *Then T' is of class $C^1(A)$ and $[T', A]^\circ$ extends from $\mathcal{D}(T')$ to a bounded form on $\mathcal{M}$, which we identify with a bounded operator $T'' = [T', A]^\circ \colon \mathcal{M} \to \mathcal{M}^*$.*

Proof. For $n \geq 0$ we have $(T' + \eta + n^2)^{-1} = ((T' + \eta)^{1/2} - \imath n)^{-1}((T' + \eta)^{1/2} + \imath n)^{-1}$, which preserves $\mathcal{D}(A)$ by assumption.

We can thus compute for $\lambda > 0$, as forms on $\mathcal{D}(A)$,

$$
\begin{aligned}
\left[(T' + \eta + n^2)^{-1}, A \right] &= \left[\left((T' + \eta)^{\frac{1}{2}} - \imath n \right)^{-1}, A \right] \left((T' + \eta)^{\frac{1}{2}} + \imath n \right)^{-1} \\
&\quad + \left((T' + \eta)^{\frac{1}{2}} - \imath n \right)^{-1} \left[\left((T' + \eta)^{\frac{1}{2}} + \imath n \right)^{-1}, A \right] \\
&= -\left((T' + \eta)^{\frac{1}{2}} - \imath n \right)^{-1} \left[(T' + \eta)^{\frac{1}{2}}, A \right]^\circ \left(T' + \eta + n^2 \right)^{-1} \\
&\quad - \left(T' + \eta + n^2 \right)^{-1} \left[(T' + \eta)^{\frac{1}{2}}, A \right]^\circ \left((T' + \eta)^{\frac{1}{2}} + \imath n \right)^{-1},
\end{aligned}
$$

which extends to a bounded operator by assumption. Hence T' is of class $C^1(A)$.

Abbreviate as in Lemma A.11, $I_{n^2}(T') = n^2 (T' + \eta + n^2)^{-1}$ and

$$T'_{n^2} = (T' + \eta) I_n(T') = n^2 \mathbb{1}_{\mathcal{H}} - n^4 \left(T' + \eta + n^2 \right)^{-1}.$$

On $\mathcal{D}(A) \cap \mathcal{D}(T')$ we compute

$$
\begin{aligned}
\left[T'_{n^2}, A\right] &= n^2\left((T'+\eta)^{\frac{1}{2}} - \imath n\right)^{-1}\left[(T'+\eta)^{\frac{1}{2}}, A\right]^{\circ} I_{n^2}(T') \\
&\quad + I_{n^2}(T')\left[(T'+\eta)^{\frac{1}{2}}, A\right]^{\circ} n^2\left((T'+\eta)^{\frac{1}{2}} + \imath n\right)^{-1} \\
&= I_{n^2}(T')\Big\{\left((T'+\eta)^{\frac{1}{2}} + \imath n\right)\left[(T'+\eta)^{\frac{1}{2}}, A\right]^{\circ} \\
&\quad + \left[(T'+\eta)^{\frac{1}{2}}, A\right]^{\circ}\left((T'+\eta)^{\frac{1}{2}} - \imath n\right)\Big\} I_{n^2}(T') \\
&= I_{n^2}(T')\Big\{(T'+\eta)^{\frac{1}{2}}\left[(T'+\eta)^{\frac{1}{2}}, A\right]^{\circ} + \left[(T'+\eta)^{\frac{1}{2}}, A\right]^{\circ}(T'+\eta)^{\frac{1}{2}}\Big\} I_{n^2}(T').
\end{aligned}
$$

Taking the limit $n \to \infty$, cf. (A.89), results in the identity

$$
\left[T', A\right] = (T'+\eta)^{\frac{1}{2}}\left[(T'+\eta)^{\frac{1}{2}}, A\right]^{\circ} + \left[(T'+\eta)^{\frac{1}{2}}, A\right]^{\circ}(T'+\eta)^{\frac{1}{2}}
$$

in the sense of forms on $\mathcal{D}(A) \cap \mathcal{D}(T')$. The result now follows since the right-hand side extends to a bounded form on $\mathcal{M}$. $\square$

As in the previous subsection, we denote by $\epsilon_0 > 0$ the constant coming from an application of Lemma 5.5.

Lemma 5.13. *For all $\epsilon \in \mathbb{R}$ and $z \in \mathbb{C}$, with $0 < |\epsilon| \le \epsilon_0$, $\mathrm{Re}(z) \in J'$ and $\epsilon\mathrm{Im}(z) > 0$, we have*

$$
\forall u \in \mathcal{D}(A) : \quad \langle u, R_\epsilon(z) T' R_\epsilon(z) u \rangle = -\langle u, \imath[R_\epsilon(z), A]u \rangle + \imath\epsilon\langle u, R_\epsilon(z), T'' R_\epsilon(z)u \rangle.
$$

Proof. For $n > \sigma$ we abbreviate $I_n(A) = \imath n(A+\imath n)^{-1}$ and $A_n = AI_n(A)$ as in Lemma A.11. Observe the identities

$$
A_n = \imath n\mathbb{1} + n^2(A+\imath n)^{-1} \quad \text{and} \quad I_n(A)^* = I_{-n}(A^*). \tag{5.87}
$$

For $u, v \in \mathcal{D}$ and $n > \sigma$ we compute using (5.87) and Remark 5.9 1

$$
\begin{aligned}
\langle I_{-n}(A^*)u, T' I_n(A)v \rangle &= -\langle u, n^2(A+\imath n)^{-1} T'(A+\imath n)^{-1}v \rangle \\
&= \langle u, \imath[T, n^2(A+\imath n)^{-1}]v \rangle \\
&= \langle u, \imath[T_\epsilon, n^2(A+\imath n)^{-1}]v \rangle - \epsilon\langle u, [T', n^2(A+\imath n)^{-1}]v \rangle \\
&= \langle u, \imath[T_\epsilon, A_n]v \rangle + \epsilon\langle u, n(A+\imath n)^{-1}[T', A]n(A+\imath n)^{-1}v \rangle \\
&= \langle u, \imath[T_\epsilon, A_n]v \rangle + \imath\langle u, I_n(A)T'' I_n(A)u \rangle.
\end{aligned}
$$

Replacing u by $R_\epsilon(z)^*u$ and v by $R_\epsilon(z)u$ we find the identity

$$
\begin{aligned}
&\langle u, R_\epsilon(z)I_n(A)T' I_n(A)R_\epsilon(z)u \rangle \\
&= -\langle u, \imath[R_\epsilon(z), A_n]u \rangle + \imath\epsilon\langle u, R_\epsilon(z)I_n(A)T'' I_n(A)R_\epsilon(z)u \rangle.
\end{aligned}
$$

Taking the limit $n \to \infty$, using (A.91) and (A.93), we arrive at the desired identity. Here we used **(M3)**, cf. Lemma 5.12, to ensure applicability of (A.93). Note that we have $\|(A+\imath n)^{-1}u\|_{\mathcal{M}^*} \le C/n$ for $n > \sigma$, which shows that $R_\epsilon(z)I_n(A)(T'+\eta)^{1/2}$ is uniformly bounded in large n. Hence

$$
\mathrm{s} - \lim_{n\to\infty} R_\epsilon(z)I_n(A)(T'+\eta)^{1/2} = R_\epsilon(z)(T'+\eta)^{1/2}
$$

and we are done. $\square$

We are now ready to give the

Proof of Theorem 5.10. Fix a $z \in \mathbb{C}$, with $\text{Re}(z) \in J'$ and $\text{Im}(z) \neq 0$. For $u \in \mathcal{D}(A)$ and $\epsilon \in \mathbb{R}$, with $\epsilon \text{Im}(z) > 0$, we define

$$F_z(\epsilon) = \langle u, R_\epsilon(z)u \rangle.$$

Using Lemma 5.13 we can compute the ϵ-derivative of F_z:

$$\frac{\mathrm{d}F_z}{\mathrm{d}\epsilon}(\epsilon) = \imath \langle u, R_\epsilon(z)T'R_\epsilon(z)u \rangle = \langle u, [R_\epsilon(z), A]u \rangle - \epsilon \langle u, R_\epsilon(z)T''R_\epsilon(z)u \rangle.$$

Using that $T'' \in \mathcal{B}(\mathcal{M}; \mathcal{M}^*)$, cf. Lemma 5.12, we arrive at the bound

$$\left| \frac{\mathrm{d}F_z}{\mathrm{d}\epsilon}(\epsilon) \right| \leq \|R_{-\epsilon}(\bar{z})u\|_\mathcal{M} \|Au\|_{\mathcal{M}^*} + \|R_\epsilon(z)u\|_\mathcal{M} \|Au\|_{\mathcal{M}^*}$$
$$+ \epsilon C \|R_{-\epsilon}(\bar{z})u\|_\mathcal{M} \|R_\epsilon(z)u\|_\mathcal{M}.$$

Here we used that $A^*u = Au$.

Before continuing we observe that $\overline{F_z}(\epsilon) = \langle R_\epsilon(z)u, u \rangle = \langle u, R_{-\epsilon}(\bar{z})u \rangle = F_{\bar{z}}(-\epsilon)$. Hence, their norms are the same. Appealing to Lemma 5.7 1 we thus get the differential inequality

$$\left| \frac{\mathrm{d}F_z}{\mathrm{d}\epsilon}(\epsilon) \right| \leq 2\epsilon^{-\frac{1}{2}}(|F_z(\epsilon)|^{\frac{1}{2}} + C\|u\|_{\mathcal{M}^*})\|Au\|_{\mathcal{M}^*} + C|F_z(\epsilon)| + C\|u\|_{\mathcal{M}^*}^2.$$

A few applications of the inequality $ab \leq (a^2 + b^2)/2$ yields

$$\left| \frac{\mathrm{d}F_z}{\mathrm{d}\epsilon}(\epsilon) \right| \leq \epsilon^{-\frac{1}{2}}C_1|F_z(\epsilon)| + \epsilon^{-\frac{1}{2}}C_2(\|u\|_{\mathcal{M}^*}^2 + \|Au\|_{\mathcal{M}^*}^2).$$

From Gronwall's inequality and a subsequent application of Lemma 5.7 we arrive at the bound

$$|F_z(\epsilon)| \leq C_3(|F_z(\epsilon_0)| + \|u\|_{\mathcal{M}^*}^2 + \|Au\|_{\mathcal{M}^*}^2)$$
$$\leq C_4(\|u\|_{\mathcal{M}^*}^2 + \|Au\|_{\mathcal{M}^*}^2),$$

where the constant C_4 does not depend on z and $u \in \mathcal{D}(A)$.

Proposition 5.8 now implies the theorem. $\qquad\square$

As a consequence of Theorem 5.10 and [60, Thm. XIII.19], keeping in mind that the compact subinterval $J' \subset J$ was arbitrary, we finally arrive at:

Corollary 5.14. *Suppose the triple of operators T, T' and A satisfies* **(M1)**–**(M4)**. *Then $\sigma_{\text{sc}}(T) \cap J = \emptyset$.*

Proof of Theorems 5.1 and 5.2: Theorems 5.1 and 5.2 follow from Theorem 5.10 and [60, Thm. XIII.19], once we have observed that **(HG2)** implies that the triple H, H' and A_δ satisfies **(M1)**–**(M4)**; and similarly that **(LG2)** implies **(M1)**–**(M4)** for the triple $\widetilde{L}_\beta, \widetilde{L}'_\beta$ and $\widetilde{A}_\delta$. Recall that $\widetilde{L}_\beta$ and L_β have the same singular continuous spectra.

Under the conditions considered, we have already established that the point spectrum of H and $\widetilde{L}_\beta$ are finite. Hence it suffices to show that $\sigma_{\text{sc}}(H) \backslash \sigma_{\text{pp}}(H) = \emptyset$ and $\sigma_{\text{sc}}(\widetilde{L}_\beta) \backslash \sigma_{\text{pp}}(\widetilde{L}_\beta) = \emptyset$.

We begin by verifying **(M1)**. From Lemma 2.2, we know that H is of class $C^1_{\mathrm{Mo}}(N)$. Hence $U_t = \mathrm{e}^{\mathrm{i}tH}$ b-preserves $\mathcal{D}(N) = \mathcal{D}(H')$. Indeed, an easy commutation argument yields $\|(N+1)(H+z)^{-1}(N+1)^{-1}\| \leq C/|\mathrm{Im}(z)|$. From this and an approximation argument using $U_t = \mathrm{s} - \lim_{n\to\infty}(1 - (\mathrm{i}tH)/n)^{-n}$ we conclude the claim. Since $\mathcal{D}(H) = \mathcal{D}(H_0)$ and $\mathcal{D}(H') = \mathcal{D}(N)$ we see that $\mathcal{C}$ is dense in $\mathcal{D}(H') \cap \mathcal{D}(H)$ with respect to the intersection topology. On $\mathcal{C}$ the commutator is

$$
\begin{aligned}
[H', H] &= -[\phi(\mathrm{i}a_{\underline{\delta}}G), H] + [\mathrm{d}\Gamma(m_{\underline{\delta}}(|k|), \phi(G)] \\
&= -[\phi(\mathrm{i}a_{\underline{\delta}}G), K \otimes \mathbb{1}_{\mathcal{F}}] + \mathrm{i}\phi(|k|a_{\underline{\delta}}G) + \mathrm{Re}\langle a_{\underline{\delta}}G, G\rangle + \mathrm{i}\phi(\mathrm{i}m_{\underline{\delta}}(|k|)G).
\end{aligned}
$$

The computation extends by continuity to $\mathcal{D}(H') \cap \mathcal{D}(H)$ and the right-hand side is $\sqrt{N}$-bounded. This is where we need the condition **(SC)**. We conclude from Proposition A.9 that H' is of class $C^1(H)$, and, together with the computation above, also **(M1)**.

Corollary 2.9 established that $\widetilde{N}$ is of class $C^1(\widetilde{L}_\beta)$. Hence the group U_t generated by $\widetilde{L}_\beta$ b-preserves $\mathcal{D}(\widetilde{N}) = \mathcal{D}(\widetilde{L}_\beta)$. As a form identity on $\widetilde{\mathcal{C}}^{\mathrm{L}}$ we have

$$
\begin{aligned}
[\widetilde{L}'_\beta, \widetilde{L}_\beta] &= -[\phi(\mathrm{i}\tilde{a}_{\underline{\delta}}\widetilde{G}_\beta), \widetilde{L}_\beta] + [\mathrm{d}\Gamma(m_{\underline{\delta}}(\omega)), \phi(\widetilde{G}_\beta)] \\
&= -[\phi(\mathrm{i}\tilde{a}_{\underline{\delta}}\widetilde{G}_\beta), L_{\mathrm{p}} \otimes \mathbb{1}_{\widetilde{\mathcal{F}}}] + \mathrm{i}\phi(\omega\tilde{a}_{\underline{\delta}}\widetilde{G}_\beta) + \mathrm{Re}\langle\tilde{a}_{\underline{\delta}}\widetilde{G}_\beta, \widetilde{G}_\beta\rangle + \mathrm{i}\phi(\mathrm{i}m_{\underline{\delta}}(\omega)\widetilde{G}\beta).
\end{aligned}
$$

The right-hand side is $\sqrt{\widetilde{N}}$-bounded, where we as above invoke **(SC)**. Since $\widetilde{\mathcal{C}}^{\mathrm{L}}$ is dense in $\mathcal{D}(\widetilde{L}'_\beta) \cap \mathcal{D}(\widetilde{L}_\beta)$, cf. Corollary 2.9, we are in a position to conclude from Proposition A.9 that $\widetilde{L}'_\beta$ is of class $C^1_{\mathrm{Mo}}(\widetilde{L}_\beta)$. Together with the computation above we get **(M1)** for the Liouvillean.

The Mourre estimate **(M2)** was establish at zero temperature by Theorem 4.7 and at positive temperature by Theorems 4.10 (high energy) and 4.15 (low temperature). Note that being away from eigenvalues, we can get rid of a compact error by passing to a smaller energy window.

That $\widetilde{L}'_\beta$ is of class $C^1(\widetilde{A}_{\underline{\delta}})$ was established in Lemma 3.4. The same proof applies to show that H' is of class $C^1(A_{\underline{\delta}})$. Since the commutator in both cases is a field operator, we conclude from Remark 5.9 3 that the first part of **(M3)** is satisfied at both zero and positive temperature. The second part of **(M3)** is automatic for self-adjoint A, that is for the Liouvillean. For both the Hamiltonian and the Liouvillean, $\mathcal{M}$ is the domain of the square root of the number operator and the adjoint of the conjugate operators $A^*_{\underline{\delta}}$ and $\widetilde{A}^*_{\underline{\delta}}$ commute with the number operator. Hence, the b-preservation part of **(M3)** is automatic.

We are left with **(M4)**, which was established for the Liouvillean in Lemma 3.2. As for H, one can proceed as in the proof of Lemma 3.2. Note that N is of class $C^1_{\mathrm{Mo}}(H)$, by an argument simpler than the one which established **(M1)** above. $\qquad\square$

We end with the following tantalizing improvement of Theorem 5.10.

Corollary 5.15. *Suppose* **(M1)**–**(M4)**. *Let* $z \in \mathbb{C}$, *with* $\mathrm{Re}(z) \in J$ *and* $\mathrm{Im}(z) \neq 0$. *The form* $M^{\frac{1}{2}}(H - z)^{-1}M^{\frac{1}{2}}$, *extends by continuity from* $\mathcal{D}_A(M^{1/2}) = \{u \in \mathcal{M} \cap \mathcal{D}(A)\,|\,M^{1/2}u \in \mathcal{D}(A)\}$ *to a bounded form on* $\mathcal{D}(A)$, *which we denote by the same expression. Furthermore, there exists* $C > 0$ *such that for all* $u \in \mathcal{D}(A)$ *we have*

$$
\sup_{\substack{z \in \mathbb{C} \\ \mathrm{Re}(z) \in J', \mathrm{Im}(z) \neq 0}} \left|\langle u, M^{\frac{1}{2}}(H - z)^{-1}M^{\frac{1}{2}}u\rangle\right| \leq C\big(\|u\|^2 + \|Au\|^2\big).
$$

Proof. From Theorem 5.10, we get for $u \in \mathcal{D}_A(M^{1/2})$ the bound

$$
\begin{aligned}
\left|\langle M^{\frac{1}{2}}u, (H-z)^{-1}M^{\frac{1}{2}}u\rangle\right| &\le C\big(\|M^{\frac{1}{2}}u\|_{\mathcal{M}^*}^2 + \|AM^{\frac{1}{2}}u\|_{\mathcal{M}^*}^2\big) \\
&\le C\big(\|u\|^2 + \|M^{-\frac{1}{2}}AM^{\frac{1}{2}}u\|^2\big).
\end{aligned}
$$

Now compute

$$
(T'+\eta)^{-\frac{1}{2}}A(T'+\eta)^{\frac{1}{2}}u = Au + (T'+\eta)^{-\frac{1}{2}}[A,(T'+\eta)^{\frac{1}{2}}]^\circ u.
$$

By **(M3)** and the boundedness of $M^{-1/2}(T'+\eta)^{1/2}$ we get the bound

$$
\left|\langle M^{\frac{1}{2}}u, (H-z)^{-1}M^{\frac{1}{2}}u\rangle\right| \le C\big(\|u\|^2 + \|Au\|^2\big).
$$

Put $I_n = I_n((T'+\eta)^{1/2}) = n((T'+\eta)^{1/2}+n)^{-1}$, using again the notation from Lemma A.11. What remains is to verify density of $\mathcal{D}_A(M^{1/2})$ in $\mathcal{D}(A)$. Let $v \in \mathcal{D}(A)$ and define $u_n = I_n v \in \mathcal{D}_A(M^{1/2})$. Here we used that if S is of class $C^1(A)$, its resolvents preserve $\mathcal{D}(A)$. We have

$$
Au_n = AI_n v = I_n Av + I_n[(T'+\eta)^{\frac{1}{2}}, A]^\circ((T'+\eta)^{\frac{1}{2}}+n)^{-1}v
$$

and

$$
[(T'+\eta)^{\frac{1}{2}}, A]^\circ((T'+\eta)^{\frac{1}{2}}+n)^{-1}v = B(T'+\eta)^{\frac{1}{2}}((T'+\eta)^{\frac{1}{2}}+n)^{-1}v.
$$

From the observation that $(T'+\eta)^{\frac{1}{2}}((T'+\eta)^{\frac{1}{2}}+n)^{-1}$ goes strongly to zero, and (A.89), we now conclude that

$$
\lim_{n\to\infty} u_n = v \quad \text{and} \quad \lim_{n\to\infty} Au_n = Av.
$$

Hence the form extends by continuity to $\mathcal{D}(A)$. $\qquad\square$

5.3 Open Problems V

We have in this section established absence of singular continuous spectrum for Pauli-Fierz Hamiltonians at zero temperature, imposing only the fairly natural **(HG2)** condition. At positive temperatures however, we have only established this fact at large energies or sufficiently low temperature. This is of course due to us only having access to positive commutator estimate in these two regimes. We do expect the result to remain valid also for large temperatures under the condition **(LG2)**.

Problem 5.1. Prove that $\sigma_{\mathrm{sc}}(L_\beta) = \emptyset$, for all β and G satisfying **(LG2)**. This of course reduces to resolving Problem 4.1.

At a first glance the improved LAP Theorem 5.15, appears to be extraordinarily useful, in particular seen from the point of view of the infrared problem. Recall that T' and the number operator are comparable objects. However, the author have not yet met an application where the ability to absorb two half-powers of the number operator into the resolvent of Hamiltonian/Liouvillean was of any importance. In case control of the number operator was needed, there were always other ways of getting it.

Problem 5.2. Identify an application where the full power of Theorem 5.15 is essentially needed.

This problem, is not so much a problem as it is a search for an application of Theorem 5.15, where Theorem 5.10 does not suffice. In the initial phase of the work resulting in the papers [19, 20], the authors thought that Theorem 5.15 would be useful in controlling the Fermi-Golden rule operator. However, under the assumptions needed to construct the Fermi-Golden rule operator, which is precisely **(HG2)** and presumably **(LG2)**, the projection onto the unperturbed eigenspaces can be shown to absorb a full power of the number operator, rendering Theorem 5.15 unnecessary. See the discussion at the end of Subsections 3.1 and 3.2.

Finally we would like to mention a problem, not directly related to Pauli-Fierz Hamiltonians, but more of a mathematical topic. The literature is abound with Limiting Absorption Principles, but the one we proved here is new, in the sense that it does not follow from an existing theorem. It is part of a family of LAP's proved under an umbrella called "singular Mourre theory" in [19], characterized by the commutator T' not being controlled in any way by the Hamiltonian T. This type of LAP goes back to Skibsted [62], with two different extensions in [28, 51]. The original LAP of Skibsted is is a special case of [28]. The LAP established here together with those of [28, 51] form a bouquet of three LAP's none of which implies another. The distinction pertains to how one deals with the double commutators $[T, T']^0$ and T''. Here, due to the fact that resolvents of the Liouvillean is of no help in bounding errors, we are forced to only make use of T' when controlling double commutators. When studying e.g. the confined Nelson model [20, 28] or AC-Stark systems with Coulomb pair-potentials [51], one is on the other hand forced to also make use of the Hamiltonian, at least partly, when controlling double commutators. This discussion serves to prepare the ground for the last problem:

Problem 5.3. Establish a MOALAP, "Mother Of All Limiting Absorption Principles", which includes the three known singular LAP's as well as the standard LAP from regular Mourre theory, see e.g. [31, 61].

A Commutator Calculus

In this appendix we recall the notion of $C^1(A)$ regularity from [27], cf. also [4]. Mostly we will recall known facts without proof, but a few considerations are new and we include proofs for these.

A.1 Bounded Operators of Class $C^1(A)$

The basic definition is the following.

Definition A.1 (The $C^1(A)$ class of bounded operators)**.** Let A be a densely defined closed operator on $\mathcal{H}$, with domain $\mathcal{D}(A)$, and $B \in \mathcal{B}(\mathcal{H})$ a bounded operator. We say that $B \in C^1(A)$ if the commutator form $[B, A]$ defined on $\mathcal{D}(A) \cap \mathcal{D}(A^*)$ extends by continuity to a bounded form on $\mathcal{H}$. We write $[B, A]^\circ \in \mathcal{B}(\mathcal{H})$ for the bounded operator representing the form.

We will use the $C^1(A)$ calculus for maximally symmetric A only, where $\mathcal{D}(A) \cap \mathcal{D}(A^*) = \mathcal{D}(A)$. Hence in the following A will always be assumed maximally symmetric, which simplifies some results. We refer the reader to [27] for the general case.

To conform with the example of Pauli-Fierz Hamiltonians we will always assume $n_+ = \dim(\ker(A^* + \imath)) = 0$, such that $\{z \in \mathbb{C} \,|\, \mathrm{Im}(z) \geq 0\} \subset \sigma(A)$. With this choice A generates a C_0-semigroup of isometries W_t:

$$\forall \psi \in \mathcal{D}(A): \quad \frac{\mathrm{d}}{\mathrm{d}t} W_t \psi = \imath A W_t \psi.$$

Recall that a C_0-semigroup is a weakly – hence strongly – continuous semigroup of bounded operators.

The first lemma establish equivalent criteria for being of class $C^1(A)$.

Lemma A.2. *Let $B \in \mathcal{B}(\mathcal{H})$. The following are equivalent.*

  (i) $B \in C^1(A)$.

 (ii) *B maps $\mathcal{D}(A)$ into itself and $AB - BA \colon \mathcal{D}(A) \to \mathcal{H}$ extends by continuity to a bounded operator on $\mathcal{H}$.*

(iii) *There exists $C > 0$ such that $\|BW_t - W_t B\| \leq Ct$, for $0 \leq t \leq 1$.*

The following lemma establishes the rules of the calculus

Lemma A.3. *Let $B, C \in \mathcal{B}(\mathcal{H})$ such that $B, C \in C^1(A)$. The following holds*

  1. *$BC \in C^1(A)$ and $[A, BC]^\circ = [A, B]^\circ C + B[A, C]^\circ$.*

  2. *If B is invertible, then $B^{-1} \in C^1(A)$ and $[B^{-1}, A]^\circ = -B^{-1}[A, B]^\circ B^{-1}$.*

  3. *If B is self-adjoint, then $\imath[B, A]^\circ$ is self-adjoint.*

  4. $\mathrm{s} - \lim_{t \to 0_+} t^{-1}(BW_t - W_t B) = \imath[B, A]^\circ$.

  5. *The set $C^1(A) \subseteq \mathcal{B}(\mathcal{H})$ is closed in the weak operator topology.*

A.2 Self-adjoint Operators of Class $C^1(A)$

Definition A.4 (The $C^1(A)$ class of self-adjoint operators). Let A be a maximally symmetric operator on $\mathcal{H}$, with domain $\mathcal{D}(A)$, and S a self-adjoint operator on $\mathcal{H}$. We say that S is of class $C^1(A)$ if there exists $z \in \rho(S)$, the resolvent set of S, such that $(S - z)^{-1} \in C^1(A)$.

We will be somewhat pedantic and say that S is of class $C^1(A)$, instead of using the notation $S \in C^1(A)$. We prefer to think of $C^1(A)$ as a subset of $\mathcal{B}(\mathcal{H})$. We remark that if S happens to be bounded, the two definitions coincide.

Lemma A.5. *Let S be self-adjoint and of class $C^1(A)$. The following holds*

1. *For all $z \in \rho(S)$ we have $(S - z)^{-1} \in C^1(A)$.*

2. *For all $z \in \rho(S)$ we have $(S - z)^{-1} \colon \mathcal{D}(A) \to \mathcal{D}(A)$.*

3. *$\mathcal{D}(S) \cap \mathcal{D}(A)$ is dense in $\mathcal{D}(S)$.*

4. *The commutator form $[S, A]$, a priori defined on $\mathcal{D}(A) \cap \mathcal{D}(S)$, extends by continuity to a bounded form on $\mathcal{D}(S)$.*

We write $[S, A]^\circ$ both for the form on $\mathcal{D}(S)$ and for the bounded operator in $\mathcal{B}(\mathcal{D}(S); \mathcal{D}(S)^*)$ representing the form. With this notation we have the important formula

$$\forall z \in \rho(S): \quad [(S - z)^{-1}, A]^\circ = -(S - z)^{-1}[S, A]^\circ (S - z)^{-1}. \tag{A.88}$$

The following theorem from [27] is crucial for the study of point spectrum and eigenstates using commutator methods. If A is self-adjoint the theorem goes back to [4, Prop. 7.2.10].

Theorem A.6 (Virial Theorem). *Let A be maximally symmetric and S self-adjoint and of class $C^1(A)$. For any eigenstate ψ of S we have $\langle \psi, [S, A]^\circ \psi \rangle = 0$.*

Proof. Let λ be the eigenvalue associated with ψ, i.e. $S\psi = \lambda\psi$. We compute using (A.88)

$$\langle \psi, [(S + \imath)^{-1}, A]^\circ \psi \rangle = -\langle (S - \imath)^{-1}\psi, [S, A]^\circ (S + \imath)^{-1}\psi \rangle = -(\lambda + \imath)^{-2}\langle \psi, [S, A]^\circ \psi \rangle.$$

The left hand side can be computed using Lemma A.3 4

$$\langle \psi, [(S + \imath)^{-1}, A]^\circ \psi \rangle = \lim_{t \to 0_+} (\imath t)^{-1} \langle \psi, ((S + \imath)^{-1} W_t - W_t (S + \imath)^{-1})\psi \rangle = 0.$$

This concludes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

A.3 The Mourre Class

Of particular interest to us is the following class of operators.

Definition A.7 (The $C^1_{\mathrm{Mo}}(A)$ class of self-adjoint operators). Let A be a maximally symmetric operator on $\mathcal{H}$, with domain $\mathcal{D}(A)$, and S a self-adjoint operator on $\mathcal{H}$. We say that S is of class $C^1_{\mathrm{Mo}}(A)$ if S is of class $C^1(A)$ and $[S, A]^\circ \in \mathcal{B}(\mathcal{D}(S); \mathcal{H})$.

The following lemma goes back to Mourre in the self-adjoint case. Mourre used a different but equivalent definition of the $C^1_{\mathrm{Mo}}(A)$ class, cf. Proposition A.9 below.

Lemma A.8. *Let S be a self-adjoint operator of class $C^1_{\mathrm{Mo}}(A)$. The following holds*

1. *$\mathcal{D}(S) \cap \mathcal{D}(A)$ is dense in both $\mathcal{D}(S)$ and $\mathcal{D}(A)$.*
2. *$\imath[S, A]^\circ$ is a symmetric operator on $\mathcal{D}(S)$.*
3. *There exists $\sigma > 0$ such that for $z \in \mathbb{C}$ with $\mathrm{Im}(z) \geq \sigma$ we have*

$$(A + z)^{-1} \colon \mathcal{D}(S) \to \mathcal{D}(S) \quad \text{and} \quad \|S(A+z)^{-1}(S+\imath)^{-1}\| \leq C/\mathrm{Im}(z).$$

4. *Suppose $S \geq 1$. Then for any $0 \leq \alpha \leq 1$, the operator S^α is of class $C^1(A)$.*

Proof. We include a proof of 3 since it is not quite standard, although our argument tracks closely that of Mourre for the self-adjoint case [52].

Compute in the sense of form on $\mathcal{H}$, for $z \in \mathbb{C}$ with $\mathrm{Im}(z) > 0$ such that $A + z$ is invertible,

$$
\begin{aligned}
(S+\imath)^{-1}(A+z)^{-1} &= (A+z)^{-1}(S+\imath)^{-1} + [(S+\imath)^{-1}, (A+z)^{-1}] \\
&= (A+z)^{-1}(S+\imath)^{-1} + (A+z)^{-1}[A, (S+\imath)^{-1}](A+z)^{-1} \\
&= (A+z)^{-1}(S+\imath)^{-1}\big(\mathbb{1}_{\mathcal{H}} + [S, A]^\circ (S+\imath)^{-1}(A+z)^{-1}\big).
\end{aligned}
$$

Since $[S, A]^\circ (S+\imath)^{-1}$ is bounded, the operator $B(z) = \mathbb{1}_{\mathcal{H}} + [S, A]^\circ (S+\imath)^{-1}(A+z)^{-1}$ is invertible, provided $\mathrm{Im}(z)$ is sufficiently large, with $\|B(z)^{-1}\|$ bounded uniformly in large $\mathrm{Im}(z)$. For such z we thus get

$$(A+z)^{-1}(S+\imath)^{-1} = (S+\imath)^{-1}(A+z)^{-1}B(z)^{-1},$$

which completes the proof of 3.

For a proof of 4, we refer the reader to (the proof of) [19, Lemma 3.1]. $\qquad \square$

To make the connection with the assumptions used by Mourre in [52] we have

Proposition A.9. *Let S be self-adjoint. The following are equivalent*

1. *S is of class $C^1_{\mathrm{Mo}}(A)$.*
2. *The following holds:*

   (a) *$W_t \colon \mathcal{D}(S) \to \mathcal{D}(S)$ and $\sup_{0 \leq t \leq 1} \|SW_t\psi\| < \infty$ for all $\psi \in \mathcal{D}(S)$.*
   (b) *$\forall \psi, \varphi \in \mathcal{D}(S) \cap \mathcal{D}(A)$: $|\langle \psi, [S, A]\varphi\rangle| \leq C\|\psi\|\|S\varphi\|$.*

Proof. To get from 1 to 2 we note first that the property 2a is called b-stability in [27], and the fact that 1 implies b-stability was established in [27, Prop. 2.34]. The bound on the commutator form is a direct consequence of the assumption on $[S, A]^\circ$ coming from the Mourre class $C^1_{\mathrm{Mo}}(A)$.

As for 2 implies 1, we note the representation formula

$$\forall z \in \mathbb{C}, \mathrm{Im}(z) > 0 : \quad (A+z)^{-1} = \int_0^\infty e^{\imath tz} W_t \, dt.$$

The property 2a together with the uniform boundedness principle implies the existence of a constant $C > 0$ such that $\|(S+\imath)W_t(S+\imath)^{-1}\| \leq C^t$, for all $t \geq 0$. This together with the representation formula implies that for $\mathrm{Im}(z) > \sigma = \ln(C)$ we have

$$(A+z)^{-1} \colon \mathcal{D}(S) \to \mathcal{D}(S).$$

Compute now for $\mu \neq 0$ and $z \in \mathbb{C}$ with $\mathrm{Im}(z) > \sigma$:

$$
\begin{aligned}
(A + z)^{-1}(S + \imath\mu)^{-1} &= (S + \imath\mu)^{-1}(A + z)^{-1} + [(A + z)^{-1}, (S + \imath\mu)^{-1}] \\
&= (S + \imath\mu)^{-1}(A + z)^{-1} + (S + \imath\mu)^{-1}[S, (A + z)^{-1}](S + \imath\mu)^{-1} \\
&= (S + \imath\mu)^{-1}(A + z)^{-1}\left(\mathbb{1}_{\mathcal{H}} - [S, A](A + z)^{-1}(S + \imath\mu)^{-1}\right).
\end{aligned}
$$

It now follows as in the proof of Lemma A.8 that for μ sufficiently large we have $(S + \imath\mu)^{-1} \colon \mathcal{D}(A) \to \mathcal{D}(A)$ and

$$
[(S + \imath\mu)^{-1}, A] = -(S + \imath\mu)^{-1}[S, A](S + \imath\mu)^{-1},
$$

a priori as a form on $\mathcal{D}(A)$, extends by continuity to a bounded for on $\mathcal{H}$. Hence S is of class $C^1(A)$. The extension property of $[S, A]^\circ$ needed to be of class $C^1_{\mathrm{Mo}}(A)$ now follows directly from 2b. $\qquad \square$

We recall the following special case of [27, Thm. 2.25], which is an extension of a result going back to [62].

Proposition A.10. *Let A be a maximally symmetric operator on $\mathcal{H}$ and S a self-adjoint operator on $\mathcal{H}$ of class $C^1_{\mathrm{Mo}}(A)$. Then the operators $T_\pm = S \pm \imath A$ defined on $\mathcal{D}(A) \cap \mathcal{D}(S)$ are closed and $T_\pm^* = T_\mp$.*

The last two pieces of information we need regarding the Mourre class are the following lemma and proposition.

Lemma A.11. *Let A be a maximally symmetric operator on $\mathcal{H}$. Define for $n \in \mathbb{N}$ bounded operators:*

$$
I_n(A) = \imath n(A + \imath n)^{-1} \quad \text{and} \quad A_n = AI_n(A).
$$

The following holds

$$
\forall u \in \mathcal{H}: \qquad\qquad \lim_{n \to \infty} I_n(A)u = u, \tag{A.89}
$$

$$
\forall u \in \mathcal{H}: \qquad\qquad \lim_{n \to \infty} I_{-n}(A^*)u = u, \tag{A.90}
$$

$$
\forall u \in \mathcal{D}(A): \qquad\qquad \lim_{n \to \infty} A_n u = Au. \tag{A.91}
$$

Suppose S is self-adjoint and of class $C^1_{\mathrm{Mo}}(A)$. Then

$$
\forall u \in \mathcal{D}(A): \qquad\qquad \lim_{n \to \infty} AI_n(S)u = Su, \tag{A.92}
$$

$$
\forall u \in \mathcal{D}(S): \qquad\qquad \lim_{n \to \infty} SI_n(A)u = Su. \tag{A.93}
$$

If $S \geq \mathbb{1}$, (A.92) also holds with $I_n(S) = n(S + n)^{-1}$.

Proof. The statement (A.89) is obvious since $I_n(A)$ is uniformly bounded and the identity holds for $u \in \mathcal{D}(A)$ by the computation $I_n(A)u = u - (A + \imath n)^{-1}Au$.

As for (A.91), it follows from the similar computation

$$
A_n u = \imath n A(A + \imath n)^{-1}u = -A(A + \imath n)^{-1}Au + Au.
$$

That the first term converges to zero again follows from $A(A + \imath n)^{-1}$ being uniformly bounded in n, and converging to zero strongly on $\mathcal{D}(A)$.

As for (A.92) and (A.93), we first remark that $I_n(S)\colon \mathcal{D}(A) \to \mathcal{D}(A)$, for all $n \geq 1$, and there exists $n_0 \in \mathbb{N}$ such that $I_n(A)\colon \mathcal{D}(S) \to \mathcal{D}(S)$, for all $n \geq n_0$. These properties follow from Lemma A.5 2 and Lemma A.8 3.

To establish (A.92) we compute for $u \in \mathcal{D}(A)$ and $n \geq 1$:

$$AI_n(S)u = \imath n A(S + \imath n)^{-1}u = I_n(S)Au + I_n(S)[S, A]^\circ(S + \imath n)^{-1}u.$$

The first term converges to Au by (A.89), and the second converges to zero. Here we again used that $\mathrm{s} - \lim_{n\to\infty} S(S + \imath n)^{-1} = 0$.

Finally, for (A.93) we compute for $u \in \mathcal{D}(S)$ and $n \geq n_0$:

$$SI_n(A)u = \imath n S(A + \imath n)^{-1}u = I_n(A)Su - I_n(A)[S, A]^\circ(A + \imath n)^{-1}u.$$

The result now follows from (A.89) and the bound in Lemma A.8 3. $\qquad\square$

Corollary A.12. *Let $S \geq 1$ be a self-adjoint operator of class $C^1_{\mathrm{Mo}}(A)$. Then, for any $0 < \alpha < 1$, $\mathcal{D}(S) \cap \mathcal{D}(A)$ is dense in $\mathcal{D}(S^\alpha) \cap \mathcal{D}(A)$ with respect to the intersection topology.*

Proof. Let $\psi \in \mathcal{D}(S^\alpha) \cap \mathcal{D}(A)$ and put $\psi_n = I_n(S)\psi \in \mathcal{D}(S) \cap \mathcal{D}(A)$, with $I_n(S) = n(S + n)^{-1}$ as in Lemma A.11. Clearly $S^\alpha \psi_n = I_n(S)S^\alpha\psi \to S^\alpha\psi$ by (A.89). That $A\psi_n \to A\psi$ follows from (A.92). This concludes the proof. $\qquad\square$

Proposition A.13. *Let $S \geq \mathbb{1}_\mathcal{H}$ be a self-adjoint operator on $\mathcal{H}$ of class $C^1_{\mathrm{Mo}}(A)$, and suppose there exists $0 \leq \rho < 1$ such that $[S, A]^\circ \in \mathcal{B}(\mathcal{D}(S^\rho); \mathcal{H})$. The following holds*

1. *For $0 < \alpha < 1$ we have $S^\alpha \in C^1_{\mathrm{Mo}}(A)$ and for $\widetilde{\rho} \in [0, 1]$, with $\widetilde{\rho} > 1 - (1-\rho)/\alpha$, we have $[S^\alpha, A]^\circ \in \mathcal{B}(\mathcal{D}(S^{\widetilde{\rho}\alpha}), \mathcal{H})$. If $\alpha < 1 - \rho$, the operator $[S^\alpha, A]^\circ$ is in particular bounded.*

2. *There exist $C > 0$ and $n \in \mathbb{N}$ such that: for any $z \in \mathbb{C}$, with $\mathrm{Im}(z) > 0$, we have $(A + z)^{-1}\colon \mathcal{D}(S) \to \mathcal{D}(S)$ and $\|S(A + z)^{-1}S^{-1}\| \leq C(1 + \mathrm{Im}(z)^n)$.*

3. *Suppose A is furthermore self-adjoint. Then, for any $f \in C^\infty(\mathbb{R})$ satisfying $\sup_{x\in\mathbb{R}} |x|^n|\mathrm{d}^n f/\mathrm{d}x^n(x)| < \infty$, we have $f(A)\colon \mathcal{D}(S) \to \mathcal{D}(S)$ continuously.*

Proof. We begin with 1. First of all, recall that by Lemma A.8 4, S^α is of class $C^1(A)$ for all $0 \leq \alpha \leq 1$.

Observe the representation formula

$$S^\alpha = c_\alpha \int_0^\infty t^\alpha(t^{-1} - (S + t)^{-1})\,\mathrm{d}t,$$

where the integral converges strongly on $\mathcal{D}(S)$. Here $c_\alpha = \sin(\alpha\pi)/\pi$. We can now compute as a form on $\mathcal{D}(S) \cap \mathcal{D}(A)$:

$$\begin{aligned} [A, S^\alpha] &= c_\alpha \int_0^\infty t^\alpha(S + t)^{-1}[S, A]^0(S + t)^\alpha \mathrm{d}t \\ &= c_\alpha \int_0^\infty t^\alpha(S + t)^{-1}BS^\rho(S + t)^\alpha \,\mathrm{d}t, \end{aligned}$$

where $B = [S, A]^\circ S^{-\rho}$ is bounded by assumption. Hence, for $\varphi, \psi \in \mathcal{D}(S) \cap \mathcal{D}(A)$ and $\widetilde{\rho} \in (0, 1)$ with $\widetilde{\rho}\alpha \leq \rho$, we have:

$$
\begin{aligned}
|\langle \varphi, [A, S^\alpha]^\circ \psi \rangle| &\leq c_\alpha \int_0^\infty t^\alpha |\langle (S+t)^{-1}\varphi, B(S+t)^{-1}S^\rho\psi \rangle| \, dt \\
&\leq c_\alpha \|B\| \int_0^\infty t^{\alpha-1}(S+t)^{-1}S^{\rho-\widetilde{\rho}\alpha)} \| \, dt \, \|\varphi\| \|S^{\widetilde{\rho}\alpha}\psi\| \\
&\leq c_\alpha \|B\| \int_0^\infty t^{\alpha-2+\rho-\widetilde{\rho}\alpha} \, dt \, \|\varphi\| \|S^{\widetilde{\rho}\alpha}\psi\|.
\end{aligned}
$$

To get something finite on the right-hand side we need to have

$$
\alpha(1 - \widetilde{\rho}) + \rho - 2 < -1.
$$

Hence we must take $\widetilde{\rho} > 1 - (1 - \rho)/\alpha$. For such $\widetilde{\rho}$ the estimate above now extends by continuity first to $\varphi, \psi \in \mathcal{D}(S)$, cf. Lemma A.5 3, and subsequently to $\varphi \in \mathcal{H}$ and $\psi \in \mathcal{D}(S^{\widetilde{\rho}\alpha})$. This completes the proof of 1.

We proceed to establish 2. Using 1 we can find a finite decreasing sequence $1 = \rho_0 > \rho_1 > \cdots > \rho_n > 0$, such that

$$
\forall 0 \leq j \leq n - 1 : \ [S^{\rho_j}, A]^\circ S^{-\rho_{j+1}} \text{ and } [S^{\rho_n}, A]^\circ \text{ are bounded.}
$$

Let $\sigma = \max_{j=0,\ldots,n} \sigma_j$, where σ_j comes from Lemma A.8 3, applied with the self-adjoint operator S^{ρ_j}. Put $B_j = [S^{\rho_j}, A]^\circ S^{-\rho_{j+1}}$, for $j \leq n - 1$ and set $B_n = [S^{\rho_n}, A]^\circ$.

We can now compute for $z \in \mathbb{C}$ with $\operatorname{Im}(z) < \sigma$ as an operator identity on $\mathcal{D}(S)$

$$
\begin{aligned}
S(A + z)^{-1} - (A + z)^{-1}S &= -(A + z)^{-1}B_0 S^{\rho_1}(A + z)^{-1} \\
&= \sum_{j=0}^{n-1}(-1)^{j+1}\big((A+z)^{-1}B_0\big)\cdots\big((A+z)^{-1}B_j\big)(A+z)^{-1}S^{\rho_{j+1}} \\
&\quad + (-1)^{n+1}\big((A+z)^{-1}B_0\big)\cdots\big((A+z)^{-1}B_n\big)(A+z)^{-1}.
\end{aligned}
$$

The right-hand side extends analytically to $z \in \mathbb{C}$, with $\operatorname{Im}(z) > 0$ and hence for all such z we have proved that $(A + z)^{-1}\colon \mathcal{D}(S) \to \mathcal{D}(S)$ and

$$
\|S(A + z)^{-1}S^{-1}\| \leq C(1 + \operatorname{Im}(z))^{-n}, \tag{A.94}
$$

where $C > 0$ does not depend on z. This proves 2. If A is self-adjoint, the conclusion extends to z with $\operatorname{Im}(z) < 0$ provided an absolute value is inserted on the right-hand side of (A.94).

The last claim 3 now follows by an almost analytic extension argument, just as in the last step of the proof of [19, Lemma 3.3]. We skip the details here. $\qquad\square$

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